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ADE spectra in conformal field theory

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Abstract

We demonstrate that certain Virasoro characters (and their linear combinations) in minimal and non-minimal conformal models which admit factorized forms are manifestly related to the ADE series. This permits to extract quasi-particle spectra of a Lie algebraic nature which resembles the features of Toda field theory. These spectra possibly admit a construction in terms of the W_n-generators. In the course of our analysis we establish interrelations between the factorized characters related to the parafermionic models, the compactified boson and the minimal models.

1. Introduction

It is well known, that a large class of off-critical integrable models is related to affine Toda field theories [1] or RSOS-statistical models [2], which possess a rich underlying Lie algebraic structure. Since these models can be regarded as perturbed conformal field theories, it is suggestive to recover the underlying Lie algebraic structure also in the conformal limit. Of primary interest is to identify the conformal counterparts of the off-critical particle spectrum. One way to achieve this is to analyze the quasi-particle spectrum, which results from certain expressions of the Virasoro characters $\chi(q)$ or their linear combinations. Hitherto this analysis was mainly performed [3] for formulae of the form

$$\chi(q) = q^{\text{const}} \sum_{\boldsymbol{l}} \frac{q^{\boldsymbol{l}^t A \boldsymbol{l} + \boldsymbol{B}\cdot\boldsymbol{l}}}{(q)_{l_1}\dots(q)_{l_r}} . \tag{1}$$

Here r is the rank of the related Lie algebra $\mathbf{g}$, the matrix A coincides with the inverse of the Cartan matrix, $\boldsymbol{B}$ characterizes the super-selection sector, $(q)_l := \prod_{k=1}^{l}(1-q^k)$, and there may be certain restrictions on the summation over $\boldsymbol{l}$. Following the prescription of [3] one can always obtain a quasi-particle spectrum once a

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character admits a representation in the form of Eq. (1). It should be noted that such spectra can not be obtained form the standard form of the Virasoro characters (10).

In the following we will demonstrate that one also recovers Lie algebraic structures in certain Virasoro characters or their linear combinations which admit the factorized form

$$\frac{q^{\text{const}}}{\{1\}_1^-}\{x_1;\ldots;x_N\}_y^-\{x'_1;\ldots;x'_M\}_y^+ , \tag{2}$$

where we adopt the notations of [4]

$$\{x\}_y^\pm := \prod_{k=0}^{\infty}(1\pm q^{x+ky}),\quad \{x_1;\ldots;x_n\}_y^\pm := \prod_{a=1}^{n}\{x_a\}_y^\pm .$$

In many cases (see [4,5] for details) expressions of the type (2) can be rewritten in the form (1), but now A is entirely absent or, at most, is a diagonal matrix. There are no restrictions on the summation over $\boldsymbol{l}$, and we allow terms of the form $(q^y)_l$ in the denominator (which may be regarded as an anionic feature [5]).

Unlike the conventional form for the Virasoro characters (10), formulae (1) and (2) allow to extract the leading order behaviour in the limit $q \to 1^-$ by means of a saddle point analysis, see e.g. [3,5]. For a slightly generalized version of (1), in the sense that all $(q)_l$ are replaced by $(q^y)_l$, this analysis leads to

$$z_i^y = \prod_{j=1}^{r}(1-z_j)^{(A_{ij}+A_{ji})},\quad c_{\text{eff}} = \frac{6}{y\pi^2}\sum_{i=1}^{r}L(z_i) . \tag{3}$$

This means solving the former set of equations for the unknown quantities z_i, we may compute the effective central charge thereafter by means of the latter equation in terms of Rogers dilogarithm $L(x)$. Recall that the effective central charge is defined as $c_{\text{eff}} = c - 24h'$, where h' is the lowest conformal weight occurring in the model. There exist inequivalent solutions to Eqs. (3) leading to the same effective central charge corresponding either to the form (1) or (2). When treating these equations as formal series, such computations give a first hint on possible candidates for characters.

Alternatively, with regard to factorization, we can exploit the essential fact that the blocks $\{x\}_y^\pm$ are closely related to the so-called quantum dilogarithm and we can easily compute their contributions to the effective central charge. As explained in Ref. [4], each block $(\{x\}_y^-)^{\pm 1}$ and $(\{x\}_y^+)^{\pm 1}$ in expressions of type (2) contributes

$$\Delta c_{\text{eff}} = \mp\frac{1}{y},\quad \text{and}\quad \Delta c_{\text{eff}} = \pm\frac{1}{2y}, \tag{4}$$

respectively. In the course of our argument, i.e. when we consider the difference of the Virasoro characters, we will also need the notion of the secondary effective central charge

$$\tilde{c} = 1 - 24h'' , \tag{5}$$

where h'' is the next to lowest conformal weight occurring in the model.

2. ADE structure

Let $\mathbf{g}$ be a Lie algebra of rank r and h be its Coxeter number. We define the following function related to $\mathbf{g}$

$$\Xi^{g}(\boldsymbol{x},q) = \frac{q^{\text{const}}}{\{x_1;\ldots;x_r\}_{\frac{h}{2}+1}^-} , \tag{6}$$

Table 1
Effective central charges for minimal affine Toda field theories

g	$A_n^{(1)}$	$D_n^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$	$A_{2n}^{(2)}$
c_{eff}	$\frac{2n}{n+3}$	1	$\frac{6}{7}$	$\frac{7}{10}$	$\frac{1}{2}$	$\frac{2n}{2n+3}$

with $\boldsymbol{x}$ obeying the condition

$$x_a + x_{r+1-a} = h/2 + 1\,, \quad a = 1,\ldots,r\,, \tag{7}$$

and for odd r we put $x_{\frac{r+1}{2}} = \frac{h}{4} + \frac{1}{2}$. Our aim is to find conformal models such that their characters or possibly linear combinations coincide with (6) for appropriately chosen q^{const} and $\boldsymbol{x}$. Such conformal models have quasi-particle spectra, generated by (6) for the related sectors, with the number of different particle species equal to the rank r.

The question arises for which conformal models can we expect (6) to be a character? Exploiting (4), we readily find the corresponding effective central charge

$$c_{\text{eff}}(\boldsymbol{g}) = \frac{2r}{h+2} = \frac{2r^2}{\dim \boldsymbol{g} + r}\,. \tag{8}$$

On the other hand, the analysis of the ultra-violet limit of the thermodynamic Bethe ansatz [6] for the ADE related minimal scattering matrices of affine Toda field theory leads to the following effective central charges (see Table 1).

Thus, we see that, upon substitution of the related Lie algebraic quantities [3] of the simply laced algebras (see e.g. [7]), Eq. (8) recovers all the effective central charges in Table 1. Furthermore it turns out that for **g** from this table corresponding to minimal models or $c = 1$ models we are always able to identify several $\Xi^{\boldsymbol{g}}(\boldsymbol{x},q)$ with single Virasoro characters or specific linear combinations of them.

In addition, there exist characters which exhibit even stronger Lie algebraic features. They are given by (6) with the values of x_a chosen as follows (which is a particular case of (7))

$$2x_a - 1 = e_a\,, \quad a = 1,\ldots,r\,, \tag{9}$$

where $\{e_a\}$ stands for the set of the exponents of the Lie algebra **g**. We denote this particular character as $\Xi^{\boldsymbol{g}}(q)$.

2.1. Minimal models

The minimal models [8] are parameterized by a pair (s,t) of co-prime positive integers and the corresponding central charge is $c(s,t) = 1 - \frac{6(s-t)^2}{st}$. Labeling the highest weights as $h_{n,m}^{s,t} = \frac{(nt-ms)^2-(s-t)^2}{4st}$, with the restrictions $1 \le n \le s-1$ and $1 \le m \le t-1$, the usual form of the characters of irreducible highest weight representations reads [9]

$$\chi_{n,m}^{s,t}(q) = \eta_{n,m}^{s,t} \sum_{k=-\infty}^{\infty} q^{stk^2}\left(q^{k(nt-ms)} - q^{k(nt+ms)+nm}\right). \tag{10}$$

Here we abbreviated the ubiquitous factor $\eta_{n,m}^{s,t} := q^{h_{n,m}^{s,t} - \frac{c(s,t)}{24}} / \{1\}_1^-$ by an analogy with the eta-function. The (secondary) effective central charge is easy to find (see e.g. [4])

$$c_{\text{eff}}(s,t) = 1 - \frac{6}{st}\,, \quad \tilde{c}(s,t) = 1 - \frac{24}{st}\,. \tag{11}$$

[3] For a twisted affine Lie algebra of type $X_N^{(k)}$ one introduces h – the Coxeter number and $h^{(k)} = kh$. We should use h in (8) for $A_{2n}^{(2)}$.

Thus, the values of $c_{\rm eff}$ which are less than 1 can be matched as follows

$$c_{\rm eff}\left(A_1^{(1)}\right)=c_{\rm eff}\left(E_8^{(1)}\right)=c_{\rm eff}(3,4)\,, \tag{12}$$

$$c_{\rm eff}\left(A_2^{(1)}\right)=c_{\rm eff}(5,6)=c_{\rm eff}(3,10)=c_{\rm eff}(2,15)\,, \tag{13}$$

$$c_{\rm eff}\left(E_6^{(1)}\right)=c_{\rm eff}(6,7)=c_{\rm eff}(3,14)=c_{\rm eff}(2,21)\,, \tag{14}$$

$$c_{\rm eff}\left(E_7^{(1)}\right)=c_{\rm eff}(4,5)\,, \tag{15}$$

$$c_{\rm eff}\left(A_{2n}^{(2)}\right)=c_{\rm eff}(2,2n+3)\,. \tag{16}$$

We see, that the matching is, in general, not unique. For (16) it depends on n – the first non-unique representations occur for $n=6$ and $n=9$ and coincide with (13) and (14), respectively. Therefore we might have for instance relations between $A_2^{(1)}\sim A_{12}^{(2)}$ and $E_6^{(1)}\sim A_{18}^{(2)}$. Some of these apparent ambiguities are easily explained as the consequence of a symmetry property of the characters. For instance we observe that Eq. (10) possesses the symmetry: $\chi_{\alpha n,m}^{\alpha s,t}(q)=\chi_{n,\alpha m}^{s,\alpha t}(q)$, for instance $\chi_{2,m}^{6,5}(q)=\chi_{1,2m}^{3,10}(q)$.

Of course Eqs. (12)–(16) are only to be understood as a first hint on a possibility for characters in the corresponding models to be of the form $\Xi^{g}(\boldsymbol{x},q)$. In order to make the identifications more precise, we have to resort to more stringent properties of the characters. We shall be using previously obtained results [4,10] on representation of characters of minimal models in the form (2). In Ref. [10] it was proven, that for $M=0$ and $x_i\neq x_j$ for $i\neq j$ in (2) the only possible factorizable single characters are

$$\chi_{n,m}^{2n,t}(q)=\eta_{n,m}^{2n,t}\{nm;nt;nt-nm\}_{nt}^{-}\,, \tag{17}$$

$$\chi_{n,m}^{3n,t}(q)=\eta_{n,m}^{3n,t}\{2nt;nm;2nt-nm\}_{2nt}^{-}\{2nt-2nm;2nt+2nm\}_{4nt}^{-}\,. \tag{18}$$

Combining now characters in a linear way, the property to be factorizable remains still exceptional. It was argued in Ref. [4] that

$$\chi_{n,m}^{s,t}(q)\pm\chi_{n,t-m}^{s,t}(q) \tag{19}$$

are the only combinations of characters in the same model which have a chance to acquire the form (2) with reasonably small N and M. The limit $q\to1^-$ of the upper and lower signs in (19) is governed by $c_{\rm eff}$ and $\tilde{c}$, respectively, which can be seen form their properties with respect to the S-modular transformation [4].

The following factorizable combinations of type (19) where found in Ref. [4]

$$\chi_{n,m}^{3n,t}(q)\pm\chi_{2n,m}^{3n,t}(q)=\eta_{n,m}^{3n,t}\{nm;nt-nm\}_{nt}^{-}\left\{\frac{nt}{2}\right\}_{\frac{nt}{2}}^{-}\left\{\frac{nt-2nm}{4};\frac{nt+2nm}{4}\right\}_{\frac{nt}{2}}^{\pm}\,, \tag{20}$$

$$\chi_{n,m}^{4n,t}(q)-\chi_{3n,m}^{4n,t}(q)=\eta_{n,m}^{4n,t}\left\{\frac{nt}{2};\frac{nt}{2}-nm;nm\right\}_{\frac{nt}{2}}^{-}\,, \tag{21}$$

$$\chi_{n,m}^{4n,t}(q)+\chi_{3n,m}^{4n,t}(q)=\eta_{n,m}^{4n,t}\{nm;nt;nt-nm\}_{nt}^{-}\left\{\frac{nt}{2}-nm;\frac{nt}{2}+nm;\frac{nt}{2}\right\}_{nt}^{+}\,, \tag{22}$$

$$\chi_{n,m}^{6n,t}(q)-\chi_{5n,m}^{6n,t}(q)=\eta_{n,m}^{6n,t}\{nt;nm;nt-nm\}_{nt}^{-}\{nt-2nm;nt+2nm\}_{2nt}^{-}\,. \tag{23}$$

Table 2
Representation of characters in the form of (6) and for differences of the type (23) in the form (24). The replacement of the blocks $\{x\}^-$ typed in bold by $\{x\}^+$ yields the corresponding differences of characters

g	sectors	$\{x_1;\ldots;x_r\}_y^-$
E_8	$\chi_{1,1}^{3,4}$	$\{2;3;4;5;11;12;13;14\}_{16}^-$
	$\chi_{1,2}^{3,4}$	$\{1;3;5;7;9;11;13;15\}_{16}^-$
	$\chi_{1,3}^{3,4}$	$\{1;4;6;7;9;10;12;15\}_{16}^-$
A_1	$\chi_{1,2}^{3,4}$	$\{1\}_2^-$
E_7	$\chi_{2,1}^{4,5}$	$\{1;3;4;5;6;7;9\}_{10}^-$
	$\chi_{2,2}^{4,5}$	$\{1;2;3;5;7;8;9\}_{10}^-$
A_2	$\chi_{1,2}^{5,6}+\chi_{1,4}^{5,6}$	$\{1;\frac{3}{2}\}_{5/2}^-$
	$\chi_{2,2}^{5,6}+\chi_{2,4}^{5,6}$	$\{\frac{1}{2};2\}_{5/2}^-$
	$\chi_{1,1}^{5,6}-\chi_{1,5}^{5,6}$	$\{2;8\}_{10}^-$
	$\chi_{2,1}^{5,6}-\chi_{2,5}^{5,6}$	$\{4;6\}_{10}^-$
E_6	$\chi_{2,1}^{6,7}+\chi_{4,1}^{6,7}$	$\{1;\boldsymbol{\frac{5}{2}};3;4;\boldsymbol{\frac{9}{2}};6\}_7^-$
	$\chi_{2,2}^{6,7}+\chi_{4,2}^{6,7}$	$\{1;\boldsymbol{\frac{3}{2}};2;5;\boldsymbol{\frac{11}{2}};6\}_7^-$
	$\chi_{2,3}^{6,7}+\chi_{4,3}^{6,7}$	$\{\boldsymbol{\frac{1}{2}};2;3;4;5;\boldsymbol{\frac{13}{2}}\}_7^-$
	$\chi_{1,1}^{6,7}-\chi_{1,6}^{6,7}$	$\{2;3;4;10;11;12\}_{14}^-$
	$\chi_{1,2}^{6,7}-\chi_{1,5}^{6,7}$	$\{1;4;6;8;10;13\}_{14}^-$
	$\chi_{1,3}^{6,7}-\chi_{1,4}^{6,7}$	$\{2;5;6;8;9;12\}_{14}^-$
G_2	$\chi_{1,2}^{3,4}$	$\{1;3\}_4^-$
F_4	$\chi_{1,1}^{2,7}$	$\{2;3;4;5\}_7^-$
	$\chi_{1,2}^{2,7}$	$\{1;3;4;6\}_7^-$
	$\chi_{1,3}^{2,7}$	$\{1;2;5;6\}_7^-$

Remarkably, as we demonstrate in Table 2, all identifications presented in (12)–(16) can be realized in terms of characters with the help of (17)–(23). In particular, we identify $\Xi^{\mathfrak{g}}(q)$ with the following characters

$$\Xi^{A_1^{(1)}}(q)=\chi_{1,2}^{3,4}(q)\,,\quad \Xi^{A_{2n}^{(2)}}(q)=\chi_{1,n+1}^{2,2n+3}(q)\,,$$
$$\Xi^{A_2^{(1)}}(q)=\chi_{1,2}^{5,6}(q)+\chi_{1,4}^{5,6}(q)=\chi_{1,2}^{3,10}(q)+\chi_{1,8}^{3,10}(q)\,,$$
$$\Xi^{E_6^{(1)}}(q)=\chi_{2,1}^{6,7}(q)+\chi_{2,6}^{6,7}(q)=\chi_{1,2}^{3,14}(q)+\chi_{1,12}^{3,14}(q)\,,$$
$$\Xi^{E_7^{(1)}}(q)=\chi_{2,1}^{4,5}(q)\,,\quad \Xi^{E_8^{(1)}}(q)=\chi_{1,3}^{3,4}(q)\,.$$

Since these identifications hint on the connection with massive models, i.e. affine Toda field theories, it is somewhat surprising that also the non-simply laced algebras G_2 and F_4 occur in Table 2 [4]. No connection is known between G_2- and F_4-affine Toda models and (3,4) and (2,7) minimal models, respectively. At present it seems to be just an intriguing coincidence.

It is interesting to notice that, as seen from Table 2, the differences of type (23) can also be of the form

$$\frac{q^{\mathrm{const}}}{\{x_1;\ldots;x_r\}_b^-}\,. \tag{24}$$

For instance, $\chi_{1,1}^{5,6}(q)-\chi_{1,5}^{5,6}(q)$ corresponds to $b=2h+4$, $x_a=2(e_a+1)$, and $\chi_{1,2}^{6,7}(q)-\chi_{1,5}^{6,7}(q)$ corresponds to $b=h+2$, $x_a=e_a+1$, where h and $\{e_a\}$ are the Coxeter number and exponents of A_2 and E_6, respectively.

[4] We thank W. Eholzer for pointing out to us that our formulae in Ref. [5] may include $\Xi^{G_2}(q)=\chi_{1,2}^{3,4}(q)$ and $\Xi^{F_4}(q)=\chi_{1,2}^{2,7}(q)$ as well.

Eq. (6) does not exhaust all manifest Lie algebraic functions in which combinations of characters of the type (19) can be represented. For instance, the following representation

$$\overline{\Xi}^{\,\boldsymbol{g}}(\boldsymbol{w},q) = q^{\mathrm{const}} \frac{\left\{\dfrac{h+2}{8}\right\}^{+}_{\frac{h+2}{4}}}{\{w_1;\ldots;w_{r-1}\}^{-}_{\frac{h}{2}+1}} \tag{25}$$

with $\boldsymbol{w}$ obeying the condition $w_a + w_{r-a} = h/2+1$ also occurs (see Table 3).

2.2. Compactified free Boson

In general the Fock space of a free boson may simply be constructed from a Heisenberg algebra and the corresponding Virasoro central charge equals $c=1$. The character of the Heisenberg module ($\hat{U}(1)$-Kac–Moody) is simply the inverse of the η-function, $q^{-1/24}/\{1\}^{-}_{1}$. When compactifying the boson on a circle of rational square radius $R=\sqrt{2s/t}$ one can associate highest weight representations of a $\hat{U}(1)_k$-Kac–Moody algebra to this theory. The $\hat{U}(1)_k$-algebra has an integer level, which is $k=st$. The corresponding characters read [7]

$$\hat{\chi}^{k}_{m}(q) = \hat{\eta}^{k}_{m} \sum_{l=-\infty}^{\infty} q^{kl^2+ml} = \hat{\eta}^{k}_{m}\{2k\}^{-}_{2k}\{k-m;k+m\}^{+}_{2k}, \tag{26}$$

where we denoted $\hat{\eta}^{k}_{m} := q^{h^k_m - \frac{1}{24}}/\{1\}^{-}_{1}$. The highest weight may take on the values $h^k_m = \frac{m^2}{4k}$ with $m=0,1,\ldots,k-1$.

As Table 1 indicates, the D_n-affine Toda models are related to compactified bosons. In order to recover the D_n-structure at the conformal level, we shall find realizations of $\Xi^{D_n}(q)$ in terms of (26). Similar to the case of minimal models, it will be helpful to study factorization of (combinations of) the Kac–Moody characters.

First, choosing the constant in (6) as $h^n_{n/2} - \frac{1}{24}$ we identify for even n

$$\Xi^{D_n}(q) = \hat{\eta}^{n}_{n/2} \frac{\{n\}^{-}_{n}}{\left\{\dfrac{n}{2}\right\}^{-}_{n}} = \hat{\eta}^{n}_{n/2}\{n\}^{-}_{n}\left\{\frac{n}{2}\right\}^{+}_{\frac{n}{2}} = \hat{\eta}^{n}_{n/2}\{2n\}^{-}_{2n}\left\{\frac{n}{2}\right\}^{+}_{n} = \hat{\chi}^{n}_{n/2}(q)\,. \tag{27}$$

Formally this expression also holds for odd n, albeit in this case the right hand side may not be interpreted as the character related to a compactified boson. Therefore we need another way to construct $\Xi^{D_n}(q)$ in case n is odd. For this purpose we consider the combinations $\hat{\chi}^{n}_{m}(q) \pm \hat{\chi}^{n}_{n-m}(q)$ which are analogues of (19). In particular, the $q \to 1^-$ limit of these sums and differences is governed by $c_{\mathrm{eff}} = c = 1$ and $\tilde{c}$. According to (5) and the possible values for the highest weights we have $\tilde{c} = 1 - 6/n$.

Table 3
Representation of characters in the form of (25). For the arguments in bold the same convention applies as in Table 2, including the numerator

$\boldsymbol{g}$	sectors	$\{w_1;\ldots;w_{r-1}\}^{-}_{y}$
A_1	$\chi^{3,4}_{1,1} + \chi^{3,4}_{1,3}$	1
E_7	$\chi^{4,5}_{1,1} + \chi^{4,5}_{1,4}$	$\{\frac{3}{2};2;\frac{7}{2};\boldsymbol{\frac{13}{2}};8;\boldsymbol{\frac{17}{2}}\}^{-}_{10}$
	$\chi^{4,5}_{1,2} + \chi^{4,5}_{1,3}$	$\{\boldsymbol{\frac{1}{2}};4;\boldsymbol{\frac{9}{2}};\boldsymbol{\frac{11}{2}};6;\boldsymbol{\frac{19}{2}}\}^{-}_{10}$

Exploiting the identity (2.31) obtained in Ref. [4], we find (for $0 < m < n/2$)

$$\hat{\chi}_m^n(q) \pm \hat{\chi}_{n-m}^n(q) = \hat{\eta}_m^n \left\{ \frac{n}{2} \right\}_{\frac{n}{2}}^{-} \left\{ \frac{n}{4} - \frac{m}{2}; \frac{n}{4} + \frac{m}{2} \right\}_{\frac{n}{2}}^{\pm} . \tag{28}$$

The counting, based on (4), gives the expected values of c and $\tilde{c}$. Notice that for the upper sign the r.h.s. can be identified as the product side of (26):

$$\hat{\chi}_m^n(q) + \hat{\chi}_{n-m}^n(q) = \hat{\chi}_{m/2}^{n/4}(q) . \tag{29}$$

Comparison with (27) then yields a formula for $\Xi^{D_n}(q)$ valid for both odd and even n

$$\Xi^{D_n}(q) = \hat{\chi}_n^{4n}(q) + \hat{\chi}_{3n}^{4n}(q) . \tag{30}$$

Finally, it is interesting to observe that, employing (17)–(20), we may express (26) and (28) entirely in terms of the minimal Virasoro characters:

$$\hat{\chi}_m^n(q) = \chi_{1,m}^{3,n}(q) \frac{\chi_{1,n}^{2,3n}(q)}{\chi_{1,m}^{2,n}(q)} , \tag{31}$$

$$\hat{\chi}_m^n(q) \pm \hat{\chi}_{n-m}^n(q) = \left(\chi_{1,m}^{3,n}(q) \pm \chi_{2,m}^{3,n}(q) \right) \frac{\chi_{1,n}^{2,3n}(q)}{\chi_{1,m}^{2,n}(q)} . \tag{32}$$

Here $n = 6l \pm 1$, $l \in N$ if we really regard all components on the r.h.s. as characters of irreducible Virasoro representations. This restriction can be omitted if we regard (17)–(20) just as formal series. With regard to the central charge Eqs. (31)-(32) imply

$$c_{\mathrm{eff}}\left(D_n^{(1)}\right) = c_{\mathrm{eff}}(3,n) + c_{\mathrm{eff}}(2,3n) - c_{\mathrm{eff}}(2,n) . \tag{33}$$

Thus, the connection with minimal models is more subtle than one would expect at first sight from a simple matching of the central charges, e.g. $c_{\mathrm{eff}}(D_n^{(1)}) = 2 c_{\mathrm{eff}}(3,4)$.

2.3. Parafermions

The $A_n^{(1)}$-series of affine Toda theories is known to be related in the ultra-violet limit (see e.g. [6]) to the Z_{n+1}-parafermions [11]. The corresponding central charge, $c(k) = 2(k-1)/(k+2)$ and characters may be obtained from the $S\hat{U}(2)_{k-1}/\hat{U}(1)$-coset, where $k = n+1$. Introducing the quantity $\Delta_{j,m}^k = j(j+1)/(k+2) - m^2/k$ the characters of the highest weight representation, which appear as branching functions in the coset, acquire the form [12]

$$\tilde{\chi}_{j,m}^k(q) = \frac{\tilde{\eta}_{j,m}^k}{\{1\}_1^-} \sum_{r,s=0}^{\infty} (-1)^{r+s} q^{rs(k+1) + \frac{r(r+1)}{2} + \frac{s(s+1)}{2}}$$
$$\times \left(q^{r(j+m)+s(j-m)} - q^{r(k+1-j-m)+s(k+1-j+m)+k+1-2j} \right), \tag{34}$$

where $\tilde{\eta}_{j,m}^k := q^{\Delta_{j,m}^k - \frac{c(k)}{24}} / \{1\}_1^-$. The labels are restricted as $-j \le m \le k-j$, $0 \le j \le k/2$ and $(j-m) \in Z$. In particular the $\tilde{\chi}_{0,m}^k(q)$ are the characters of the parafermionic currents ψ_m^k. The characters possess the symmetries

$$\tilde{\chi}_{j,m}^k(q) = \tilde{\chi}_{j,-m}^k(q) = \tilde{\chi}_{k/2-j,k/2-m}^k(q) . \tag{35}$$

From our observations made above for characters of the ADE related conformal models one may expect that expressions (34) exhibit A_n-type structures (e.g. possess n quasi particles and moreover acquire the form of the type $\Xi^{A_n}(q)$) in some of the cases when they admit a factorized form. We shall now discuss this issue in detail for several of the lowest ranks.

As we have seen above, factorization of linear combinations of characters occurs usually only for the specific type of combinations. Now the analogue of (19) is $\tilde{\chi}^{k}_{j,m}(q) \pm \tilde{\chi}^{k}_{j,2k-m}(q)$. One expects that the $q \to 1^-$ limit of these sums and differences is governed by $c_{\text{eff}} = c(k)$ and $\tilde{c}$, respectively. Since $h'' = \Delta^{k}_{1,1}$, Eq. (5) yields $\tilde{c} = c(k)(k-6)/k$. This is confirmed by all the examples given below.

For the parafermionic formulae (34) we do not have such powerful analytical tools (analogues to the factorization formulae (17)-(23)) at hand as in the case of the minimal models. Therefore, as a first step, we resort to an analysis with Mathematica. Typically we expand the characters up to q^{100}.

$\underline{A_1}$: In this case there are only three distinct (up to the symmetries (35)) characters and they can be matched with those of the (3,4) minimal model:

$$\tilde{\chi}^{2}_{0,0}(q) = \chi^{3,4}_{1,1}(q)\,, \quad \tilde{\chi}^{2}_{0,1}(q) = \chi^{3,4}_{1,3}(q)\,, \tag{36}$$

$$\tilde{\chi}^{2}_{\frac{1}{2},\frac{1}{2}}(q) = \chi^{3,4}_{1,2}(q) = \Xi^{A_1}(q)\,. \tag{37}$$

Thus, all the characters in this case are factorizable and moreover $\Xi^{A_1}(q)$ is present among them.

$\underline{A_2}$: There are four distinct characters in this case and they can be matched with those of the (5,6) minimal model:

$$\tilde{\chi}^{3}_{0,0}(q) = \chi^{5,6}_{1,1}(q) + \chi^{5,6}_{1,5}(q)\,, \quad \tilde{\chi}^{3}_{0,1}(q) = \chi^{5,6}_{1,3}(q)\,, \tag{38}$$

$$\tilde{\chi}^{3}_{\frac{1}{2},\frac{1}{2}}(q) = \chi^{5,6}_{2,3}(q)\,, \quad \tilde{\chi}^{3}_{\frac{1}{2},\frac{3}{2}}(q) = \chi^{5,6}_{2,1}(q) + \chi^{5,6}_{2,5}(q)\,. \tag{39}$$

Only $\tilde{\chi}^{3}_{0,1}(q)$ and $\tilde{\chi}^{3}_{\frac{1}{2},\frac{1}{2}}(q)$ are factorizable (see subsection II.A).

$\underline{A_3}$: Since $c=1$, it is suggestive to try to relate the characters to those of the compactified bosons. This turns out to be possible for all the characters (thus factorizability is guaranteed):

$$\tilde{\chi}^{4}_{0,1}(q) = \hat{\chi}^{12}_{6}(q)\,, \quad \tilde{\chi}^{4}_{0,0}(q) + \tilde{\chi}^{4}_{0,2}(q) = \hat{\chi}^{3}_{0}(q)\,, \tag{40}$$

$$\tilde{\chi}^{4}_{1,0}(q) = \hat{\chi}^{3}_{2}(q)\,, \quad \tilde{\chi}^{4}_{1,1}(q) = \hat{\chi}^{3}_{1}(q)\,, \tag{41}$$

$$\tilde{\chi}^{4}_{\frac{1}{2},\frac{1}{2}}(q) = \hat{\chi}^{4}_{1}(q)\,, \quad \tilde{\chi}^{4}_{\frac{1}{2},\frac{3}{2}}(q) = \hat{\chi}^{4}_{3}(q)\,. \tag{42}$$

Furthermore, some of linear combinations can be expressed in terms of the characters of the (3,4) minimal model (notice that $c=1$, $\tilde{c} = -1/2$ for A_3 and $c = 1/2$, $\tilde{c} = -1$ for the (3,4) minimal model):

$$\tilde{\chi}^{4}_{0,0}(q) - \tilde{\chi}^{4}_{0,2}(q) = \left(\chi^{3,4}_{1,2}(q)\right)^{-1}, \tag{43}$$

$$\tilde{\chi}^{4}_{\frac{1}{2},\frac{1}{2}}(q) \pm \tilde{\chi}^{4}_{\frac{1}{2},\frac{3}{2}}(q) = \left(\chi^{3,4}_{1,1}(q) \mp \chi^{3,4}_{1,3}(q)\right)^{-1}. \tag{44}$$

$\underline{A_4}$: No characters or linear combinations factorize.

$\underline{A_5}$: Several characters and combinations are factorizable and can be expressed via those of the (3,4) minimal model and D_n, for instance

$$\tilde{\chi}^{6}_{\frac{3}{2},\frac{1}{2}}(q) = \chi^{3,4}_{1,2}(q)\left(\hat{\chi}^{24}_{6}(q) - \hat{\chi}^{24}_{18}(q)\right),$$

$$\tilde{\chi}^{6}_{\frac{1}{2},\frac{3}{2}}(q) = \chi^{3,4}_{1,2}(q)\left(\hat{\chi}^{24}_{8}(q) - \hat{\chi}^{24}_{16}(q)\right),$$

$$\tilde{\chi}^{6}_{0,1}(q) \pm \tilde{\chi}^{6}_{0,2}(q) = \left(\chi^{3,4}_{1,1}(q) \pm \chi^{3,4}_{1,3}(q)\right)\left(\hat{\chi}^{24}_{9}(q) \mp \hat{\chi}^{24}_{15}(q)\right),$$

$$\tilde{\chi}^{6}_{1,1}(q) \pm \tilde{\chi}^{6}_{1,2}(q) = \left(\chi^{3,4}_{1,1}(q) \pm \chi^{3,4}_{1,3}(q)\right)\left(\hat{\chi}^{24}_{3}(q) \mp \hat{\chi}^{24}_{21}(q)\right).$$

Also we notice that $\tilde{\chi}^{6}_{\frac{1}{2},\frac{1}{2}}(q) - \tilde{\chi}^{6}_{\frac{1}{2},\frac{5}{2}}(q) = q^{5/96}$. Such an identity can occur only in this parafermionic model since it requires $\tilde{c} = 0$.

$\underline{A_6}$: – no combinations factorize and the only factorizable single characters are

$$\tilde{\chi}^{7}_{1,m}(q) = \tilde{\eta}^{7}_{1,m} \frac{\{3\}^{-}_{3}\,\{m;7-m;7\}^{-}_{7}}{\{1\}^{-}_{1}\,\{3m;21-3m\}^{-}_{21}}\,, \quad m = 1,2,3. \tag{45}$$

Summarizing these data, we see that, apart from the A_1 case, none of the factorizable (combinations of) characters provided by Eq. (34) for A_n can be identified as $\Xi^{A_n}(\boldsymbol{x},q)$. However, it is plausible to speculate that in general the $\Xi^{A_n}(\boldsymbol{x},q)$ might be identifiable as characters of other conformal models having the central charge $2n/(n+3)$. This conjecture is supported by the A_2 case (see Table 2) and A_3 case, in which we can identify

$$\Xi^{A_3}(q) = \hat{\chi}_3^{12}(q) + \hat{\chi}_9^{12}(q)\,. \tag{46}$$

To conclude the discussion on factorizable parafermionic characters, we notice an intriguing fact – some characters in the A_7 case exhibit an E_7 structure (cf. Tables 2 and 3):

$$\tilde{\chi}_{0,1}^{8}(q) + \tilde{\chi}_{0,3}^{8}(q) = \left(\chi_{2,1}^{4,5}(q)\right)^2,$$

$$\tilde{\chi}_{1,1}^{8}(q) + \tilde{\chi}_{1,3}^{8}(q) = \left(\chi_{2,2}^{4,5}(q)\right)^2,$$

$$\tilde{\chi}_{0,0}^{8}(q) \pm 2\tilde{\chi}_{0,2}^{8}(q) + \tilde{\chi}_{0,4}^{8}(q) = \left(\chi_{1,1}^{4,5}(q) \pm \chi_{1,4}^{4,5}(q)\right)^2,$$

$$\tilde{\chi}_{1,0}^{8}(q) \pm 2\tilde{\chi}_{1,2}^{8}(q) + \tilde{\chi}_{1,4}^{8}(q) = \left(\chi_{1,2}^{4,5}(q) \pm \chi_{1,3}^{4,5}(q)\right)^2.$$

This is the first case in which we have to combine three characters in order to obtain a factorized form. A more detailed account on the factorization of A_n-related characters will be presented elsewhere.

3. One particle states

The functions $\{x\}_y^+$ and $\frac{1}{\{x\}_y^-}$ can be written as double series in q with coefficients being $\mathscr{P}(n,m)$ (or $\mathscr{Q}(n,m)$) – the number of partitions of an integer $n \geq 0$ into m distinct (or smaller than $m+1$) non-negative integers (see e.g. [13,4]).

Applying this fact to a character of the type (2) with $x_i \neq x_j$, we obtain it in the form of a series $\chi(q) = \sum_{k=0}^{\infty} \mu_k q^k$, where the level k admits the partitioning, $k = \sum_a \sum_{i_a} p_a^{i_a}$, into parts of a specific form (e.g. (47) and (48) below). The interpretation of the $p_a^{i_a}$ as momenta of massless particles gives rise to a quasi-particle picture (developed originally for characters of the form (1) in Ref. [3]), where a character is regarded as a partition function, $\chi(q) = \sum_k \mu_k \mathrm{e}^{-\beta E_k}$. Here $q = \mathrm{e}^{-2\pi\beta v/L}$, with v being the speed of sound, and L – the size of the system. A quasi-particle spectrum constructed in this way is in one-to-one correspondence to the corresponding irreducible representations of the Virasoro algebra or some modules related to linear combinations. It is crucial to stress that this procedure is not applicable to the standard representation of the characters (i.e. of the type (10)) and is a very specific feature of the representations (1) and (2). Note that the modules which are of the form (1) do in general (if they do, they give rise to Rogers–Ramanujan type identities [4]) not factorize, such that the spectra related to (2) do not only differ in nature from the ones obtainable from (1), but are also related to different sectors.

As just explained, a quasi-particle representation can be constructed for any factorizable character of the type (2) provided that $x_i \neq x_j$. For instance, the characters (27) related to a compactified boson admit a representation with $(2k+1)$ particles. However, since we are particularly interested in spectra with Lie algebraic features, it is most natural to perform the quasi-particle analysis for the characters which admit the form $\Xi^{g}(\boldsymbol{x},q)$. In this way we obtain the following fermionic spectrum (if we employ the series involving $\mathscr{P}(n,m)$) in the units of $2\pi/L$

$$p_a^i(\boldsymbol{m}) = x_a + \left(\frac{h}{4} + \frac{1}{2}\right)(1-m_a) + \left(\frac{h}{2} + 1\right)N_a^i\,, \tag{47}$$

Table 4
Bosonic spectrum for $\chi^{5,6}_{1,2}(q)+\chi^{5,6}_{1,4}(q)$. k denotes the level and μ_k its degeneracy

k	μ_k	$p^i_1 = 1+\frac{5i}{2},\ p^i_2 = \frac{3}{2}+\frac{5i}{2}$
$\frac{1}{2}$	0	--
1	1	$\vert p^0_1\rangle$
$\frac{3}{2}$	1	$\vert p^0_2\rangle$
2	1	$\vert p^0_1,p^0_1\rangle$
$\frac{5}{2}$	1	$\vert p^0_1,p^0_2\rangle$
3	2	$\vert p^0_1,p^0_1,p^0_1\rangle$, $\vert p^0_2,p^0_2\rangle$
$\frac{7}{2}$	2	$\vert p^0_1,p^0_1,p^0_2\rangle$, $\vert p^1_1\rangle$
4	3	$\vert p^0_1,p^0_1,p^0_1,p^0_1\rangle$, $\vert p^0_1,p^0_2,p^0_2\rangle$, $\vert p^1_2\rangle$

or bosonic spectrum (if we use the series with $\mathscr{O}(n,m)$)

$$p^i_a = x_a + \left(\frac{h}{2}+1\right) N^i_a \,. \tag{48}$$

Here $\boldsymbol{x},\boldsymbol{m}$ and $\boldsymbol{N}$ parameterize the possible states. In Eq. (47) the numbers N^i_a are distinct positive integers such that $\sum_{i=1}^{m_a} N^i_a = N_a$, whereas in Eq. (48) they are arbitrary non-negative integers. Notice that for the combination of characters the levels may be half integer graded, such that also the momenta take on half integer values in this case. A sample spectrum is presented in Table 4 which illustrates how the available momenta of the form (48) are to be assembled in order to represent a state at a particular level.

Naturally the questions arise if we can interpret these spectra more deeply and if we can possibly find alternative representations for the related modules. First of all we should give a meaning to the particular combinations which occur in our analysis. In Ref. [4] we provided several possibilities. In particular the combination $\chi^{5,6}_{1,2}(q)+\chi^{5,6}_{1,4}(q)$ is of interest in the context of boundary conformal field theories, since this combination of characters coincides with the partition function $Z_{A,F}$ for the critical 3-state Potts model with boundaries [14] (F denotes the free boundary condition). It is intriguing that this combination possesses a manifestly Lie algebraic quasi-particle spectrum. The combination $\chi^{5,6}_{2,2}(q)+\chi^{5,6}_{2,4}(q)$, which coincides with $Z_{BC,F}$ in the same model possesses a slightly weaker relation to A_2.

To answer the question concerning possible representations, we recall the fact that the fields corresponding to the highest weight states satisfy the quantum equation of motion of Toda field theory [15]. It is therefore very suggestive to try to identify the presented spectra in terms of the W-algebras [16]. For $\Xi^{\mathfrak{g}}(q)$ we can make this more manifest. Changing the units of the momenta to π/L, we obtain from (48)

$$p^i_a = e_a + 1 + (h+2) N^i_a \,. \tag{49}$$

Here e_a belongs to the exponents of the Lie algebra. Since the generators of the W-algebras W_{s+1} are graded by the exponents plus one [17], we may associate the following generators to this quasi-particle spectrum

$$p^i_a \sim W_a (W_a W_{r-a})^{N^i_a} \,. \tag{50}$$

In particular, the critical 3-state Potts model with boundaries would be related to the W_3-algebra. We leave it for the future to investigate this conjecture in more detail.

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