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Nuclear Physics B 636 [FS] (2002) 611–631

NUCLEAR PHYSICS B

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Finite temperature correlation functions from form factors

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Received 25 March 2002; received in revised form 13 May 2002; accepted 22 May 2002

Abstract

We investigate proposals of how the form factor approach to compute correlation functions at zero temperature can be extended to finite temperature. For the two-point correlation function we conclude that the suggestion to use the usual form factor expansion with the modification of introducing dressing functions of various kinds is only suitable for free theories. Dynamically interacting theories require a more severe change of the form factor program. © 2002 Elsevier Science B.V. All rights reserved.

PACS: 11.10.Kk; 11.55.Ds

1. Introduction

The computation of correlation functions is one of the central objectives in quantum field theories. In general, this can only be achieved by means of perturbation theory in the coupling constant. Nonetheless, in $1+1$ space–time dimensions many exact results are known, in particular, at zero temperature. At present, one of the most successful approaches in this direction is the form factor program. Originally this method was developed to compute correlation functions for massive models at zero temperature [1,2]. More recently, it has also been demonstrated, that the approach can be employed successfully for the computation of massless correlation functions [3] for vanishing temperature. However, in a realistic set-up of a physical experiment one needs to know such functions in the finite temperature regime. Computing for instance physical quantities from linear response theory one requires the response function in form of the canonical two-point correlation

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PII: S0550-3213(02)00409-1

function at finite temperature

$$\big\langle \Delta\mathcal{O}(x,t)\Delta\mathcal{O}'\big(x',t'\big)\big\rangle_T, \tag{1.1}$$

where $\Delta\mathcal{O}(x,t) = \mathcal{O}(x,t) - \langle\mathcal{O}(x,t)\rangle_T$. For the static case, e.g., for the computation of electric and magnetic susceptibilities this goes back to [4]. Taking for instance the local operators $\mathcal{O}(x,t)$ and $\mathcal{O}'(x,t)$ to be the current $J(x,t)$, the dynamical response is needed to compute the conductivity by means of the celebrated Kubo formula [5].

In [6] a proposal was made to adapt the form factor approach to finite temperature computations. In there it was demonstrated for some operators of the Ising model that the modified approach indeed reproduces the expected results for the temperature dependent correlation functions, even when boundaries are included. In order to make the method meaningful several technical assumptions were required to eliminate various infinities. Since not all of them can be justified in an entirely rigorous fashion, further evidence is desirable to support the working of the proposed prescription, even for the Ising model. The proposal is very appealing, since apart from the introduction of a dressing function, the main structures of the $(T = 0)$-form factor approach are perpetuated. It was argued in [6] for the Ising model, that the dressing functions admit an interpretation in terms of density distribution functions. This observation was taken up in [7] and it was conjectured that the interpretation should also hold for interacting theories. Some checks which support the validity of this conjecture for the one-point function were presented in [7]. Shortly afterwards, doubts were raised in [8] on the working of these formulae for the two-point function, albeit only a counter example which required a chemical potential was provided. An additional controversy arose thereafter about the nature of the “dressing function” (see Eq. (2.15)) which has to be employed in the context of the one-point function. In [9], it was proposed to employ the on-shell free energies, rather than the pseudo-energies obtainable from the thermodynamic Bethe ansatz (TBA) as suggested in [7]. For the one-point function evidence was provided in [10] that the proposal in [9] appears to be incorrect. No explicit claims concerning the two-point functions were made in [9].

Alternatively one may also compare physical quantities which on one hand involve the temperature dependent two-point correlation functions and on the other can be computed by different means. This provides indirectly information on the temperature dependent correlation function. In such a context one can exploit the fact that one has additional parameters available in which one can develop. For instance in [11] a proposal was analyzed of how a low frequency expansion can be implemented in the calculation of the conductance in a boundary problem, by replacing the reflection amplitudes by a renormalized counterpart. Or, viewing this alternatively, the zero temperature form factors have been replaced by renormalized ones. However, it is not clear how this procedure can be formulated outside the mentioned context, in particular when no boundary is present. Also in a physical context, namely, by comparing two ways of computing susceptibilities the dressed form factor approach, similar to the one in [6,7], was tested in [12] for nondiagonal theories in the low temperature regime.

In the light of the above statements it is highly desirable to check the dressed form factor approach directly outside any physical context for the full temperature regime. So far the only few explicit computations using the dressed form factor approach may be found in [6,8]. It is, therefore, very suggestive to verify it for a simple dynamically interaction theory. The main purpose of this manuscript is to provide such an example and contribute to the above mentioned debate providing further evidence for the (non)-validity of the various proposals. In regard to the importance of (1.1), we want to focus especially on the study of the expressions for the two-point functions.

Our manuscript is organized as follows: in Section 2 we outline the various proposals made so far to evaluate temperature dependent correlation functions in the massive as well as in the massless regime. We investigate these proposals for two models: the complex free Fermion/Federbush model (Section 3) and the scaling Yang–Lee model (Section 4). In Section 5 we state our conclusions. In Appendix A we assemble various properties of functions which occur throughout our computations.

2. Temperature dependent correlation functions

We start by providing a concise review of the main features of the prescription to compute temperature dependent n-point correlation functions by means of form factors. In general a temperature state is described by a density matrix ρ and the expectation value for observables is the trace over the product of this matrix with the observables. Taking $|\psi\rangle$ to be eigenstates of the Hamiltonian H with eigenvalues E_ψ, and as usual $\beta = 1/kT$ with k being Boltzmann's constant and T the absolute temperature, the temperature dependent n-point function for the observables $\mathcal{O}_1 \cdots \mathcal{O}_n$ is defined as

$$\begin{aligned}&\big\langle \mathcal{O}_1(x_1,t_1)\cdots\mathcal{O}_n(x_n,t_n)\big\rangle_T \\ &\quad := \frac{1}{Z}\sum_{\psi} e^{-\beta E_\psi}\langle\psi|\mathcal{O}_1(x_1,t_1)\cdots\mathcal{O}_n(x_n,t_n)|\psi\rangle. \end{aligned} \tag{2.1}$$

As usual, this expression is normalized by dividing with the partition function

$$Z = \sum_{\psi} e^{-\beta E_\psi}\langle\psi|\psi\rangle, \tag{2.2}$$

which ensures that the trace over the density matrix $\rho = e^{-\beta H}/Z$ is one. The central idea of the "dressed" form factor program is now, as in the finite temperature case [1,2], to reduce the computation of the expansion (2.1) to the computation of form factors

$$F_\psi^{\mathcal{O}} = \langle\mathcal{O}(0)|\psi\rangle. \tag{2.3}$$

This is achieved simply by the insertion of $(n-1)$ complete states $\sum_\psi |\psi\rangle\langle\psi| = 1$. Suppressing for compactness the space–time dependence of the operators, this reads

$$\begin{aligned}&\langle\mathcal{O}_1\cdots\mathcal{O}_n\rangle_T \\ &\quad = \frac{1}{Z}\sum_{\psi_0\cdots\psi_{n-1}} e^{-\beta E_{\psi_0}}\langle\psi_0|\mathcal{O}_1|\psi_1\rangle\langle\psi_1|\mathcal{O}_2|\psi_2\rangle\cdots\langle\psi_{n-1}|\mathcal{O}_n|\psi_0\rangle. \end{aligned} \tag{2.4}$$

Thereafter, one needs a meaningful prescription to relate matrix elements of the form $\langle\psi'|\mathcal{O}|\psi\rangle$ to those where the vacuum is on the left, as in (2.3), and a shift operator

$$\omega_{t,x}\mathcal{O}(x,t) = f(x,t)\mathcal{O}(0)$$

which moves the operator to the origin. In the following we will take $x^\mu = (-ir, 0)$, which implies the restriction $r < \beta$ in order to ensure that $\exp(-(\beta + r)H)$ is a trace class operator.

Let us now specify the states $|\psi\rangle$ to be multi-particle states of the form

$$|\psi\rangle = \left|Z^\dagger_{\mu_1}(\theta_1) Z^\dagger_{\mu_2}(\theta_2) \cdots Z^\dagger_{\mu_n}(\theta_n)\right\rangle, \tag{2.5}$$

where the operators $Z^\dagger_\mu(\theta)$ are creation operators for a particle of type μ as a function of the rapidity θ. These operators are assumed to satisfy the Faddeev–Zamolodchikov algebra [13]

$$Z^\dagger_i(\theta_1) Z^\dagger_j(\theta_2) = S^{kl}_{ij}(\theta_{12}) Z^\dagger_k(\theta_2) Z^\dagger_l(\theta_1)$$

with S being the scattering matrix depending on the rapidity difference $\theta_{12} = \theta_1 - \theta_2$. The prescription $\langle\psi'|\mathcal{O}|\psi\rangle \to \langle\mathcal{O}|\psi\psi'\rangle$ then reads

$$\begin{aligned} &\left\langle Z_{\mu_1}\left(\theta'_1\right) \cdots Z_{\mu_n}\left(\theta'_n\right)\middle|\mathcal{O}(x,t)\middle|Z^\dagger_{\mu_n}(\theta_n) \cdots Z^\dagger_{\mu_1}(\theta_1)\right\rangle \\ &\quad = \sum_{\text{all contractions}} \left\langle \mathcal{O}(x,t) \;\middle|\; Z^\dagger_{\mu_n}(\theta_n) \cdots Z^\dagger_{\mu_1}(\theta_1) Z_{\bar{\mu}_1}\left(\theta'^-_1\right) \cdots Z_{\bar{\mu}_n}\left(\theta'^-_n\right)\right\rangle \end{aligned} \tag{2.6}$$

with $\theta^- = \theta - i\pi + i\epsilon$ and ϵ is an infinitesimal quantity and $\bar{\mu}$ being the antiparticle of μ. After shifting the operator to the origin, the r.h.s. of (2.6) involves the n-particle form factors, which we denote as

$$F_n^{\mathcal{O}|\mu_1\cdots\mu_n}(\theta_1, \ldots, \theta_n) \equiv \left\langle \mathcal{O}(0) \;\middle|\; \mathcal{Z}^\dagger_{\mu_1}(\theta_1) \cdots Z^\dagger_{\mu_n}(\theta_n)\right\rangle. \tag{2.7}$$

These form factors have to satisfy various properties [1,2], such as Watson's equations

$$\begin{aligned} &F_n^{\mathcal{O}|\cdots\mu_i\mu_{i+1}\cdots}(\ldots, \theta_i, \theta_{i+1}, \ldots) \\ &\quad = F_n^{\mathcal{O}|\cdots\mu_{i+1}\mu_i\cdots}(\ldots, \theta_{i+1}, \theta_i, \ldots) S_{\mu_i\mu_{i+1}}(\theta_{i,i+1}), \end{aligned} \tag{2.8}$$

$$\begin{aligned} &F_n^{\mathcal{O}|\mu_1\cdots\mu_n}(\theta_1 + 2\pi i, \ldots, \theta_n) \\ &\quad = \gamma^{\mathcal{O}}_{\mu_1} F_n^{\mathcal{O}|\mu_2\cdots\mu_n\mu_1}(\theta_2, \ldots, \theta_n, \theta_1), \end{aligned} \tag{2.9}$$

Lorentz invariance

$$F_n^{\mathcal{O}|\mu_1\cdots\mu_n}(\theta_1, \ldots, \theta_n) = e^{s\lambda} F_n^{\mathcal{O}|\mu_1\cdots\mu_n}(\theta_1 + \lambda, \ldots, \theta_n + \lambda), \tag{2.10}$$

and the so-called kinematic residue equation

$$\begin{aligned} &\operatorname{Re} s_{\bar{\theta}\to\theta_0} F_{n+2}^{\mathcal{O}|\bar{\mu}\mu\mu_1\cdots\mu_n}(\bar{\theta} + i\pi, \theta_0, \theta_1, \ldots, \theta_n) \\ &\quad = i\left[1 - \gamma^{\mathcal{O}}_\mu \prod_{l=1}^{n} S_{\mu\mu_l}(\theta_{0l})\right] F_n^{\mathcal{O}|\mu_1\cdots\mu_n}(\theta_1, \ldots, \theta_n). \end{aligned} \tag{2.11}$$

In (2.10) s is the Lorentz spin of $\mathcal{O}$ and $\lambda \in \mathbb{C}$ an arbitrary shift. In Eqs. (2.9) and (2.11) $\gamma^{\mathcal{O}}_\mu$ is the factor of local commutativity defined through the equal time exchange relation of the local operator $\mathcal{O}(x)$ and the field $\mathcal{O}_\mu(y)$ associated to the particle creation operators $Z^\dagger_\mu(\theta)$, i.e., $\mathcal{O}_\mu(x)\mathcal{O}(y) = \gamma^{\mathcal{O}}_\mu \mathcal{O}(y)\,\mathcal{O}_\mu(x)$ for $x^1 > y^1$. It is the singularity $\lim_{\epsilon\to 0} Z^\dagger_\mu(\theta) Z_{\bar{\mu}}(\theta^-) \to \infty$, implicit in (2.11), which is the reason for the presence of the

ϵ in (2.6). The renormalization prescription which eliminates these divergencies is outlined in [14] (see also [6,7]). Other types of singularities arise from the contractions of terms like $Z_\mu(\theta)Z_\mu^\dagger(\theta)Z_\nu(\theta')Z_\nu^\dagger(\theta') \to \delta^2(0)$. As demonstrated explicitly in [6] such terms are absorbed in the (re)normalization factor Z. In this way the one-point function [6–10,14]

$$\langle \mathcal{O}(r)\rangle_T = \frac{1}{Z}\sum_{n=0}^{\infty}\sum_{\mu_1\cdots\mu_n}\int \frac{d\theta_1\cdots d\theta_n}{n!(2\pi)^n}\exp\bigl\{-\beta\bigl[\varepsilon_{\mu_1}(\theta_1)+\cdots\varepsilon_{\mu_n}(\theta_n)\bigr]\bigr\}$$
$$\times \bigl\langle Z_{\mu_1}(\theta_1)\cdots Z_{\mu_n}(\theta_n)\bigl|\mathcal{O}(r)\bigr|Z_{\mu_n}^\dagger(\theta_n)\cdots Z_{\mu_1}^\dagger(\theta_1)\bigr\rangle \quad r<\frac{1}{T}, \tag{2.12}$$

becomes a meaningful expression. In many cases this function is an important normalization factor, however, for the reasons mentioned in the introduction we will focus our attention on the two-point function. It results to [6]

$$\bigl\langle \mathcal{O}(r)\mathcal{O}'(0)\bigr\rangle_T$$
$$=\sum_{n=1}^{\infty}\sum_{\mu_1\cdots\mu_n}\int \frac{d\theta_1\cdots d\theta_n}{n!(2\pi)^n}\prod_{i=1}^{n}\bigl[f_{\mu_i}(\theta_i,T)e^{-rT\varepsilon_{\mu_i}(\theta_i,T)}\bigr]$$
$$\times F_n^{\mathcal{O}|\mu_1\cdots\mu_n}(\theta_1,\ldots,\theta_n)\bigl[F_n^{\mathcal{O}'|\mu_1\cdots\mu_n}(\theta_1,\ldots,\theta_n)\bigr]^* \quad r<\frac{1}{T}. \tag{2.13}$$

This formula requires several explanations and comments: we dropped here as usual another infinity coming from $n=0$ in the infinite sum. The sum over the μ extends over particles and holes. This is understood in the way that to each particle type present at zero temperature one may associate a hole, such that each term at zero temperature which is summed over n-particles is mapped into 2^n-terms at finite temperature. According to [6], the form factors involving holes may be constructed from the ones of particles by an $i\pi$-shift

$$F_n^{\mathcal{O}|\mu_1\cdots\text{particle}\cdots\mu_n}(\theta_1,\ldots,\theta_i,\ldots,\theta_n)$$
$$=F_n^{\mathcal{O}|\mu_1\cdots\text{hole}\cdots\mu_n}(\theta_1,\ldots,\theta_i-i\pi,\ldots,\theta_n). \tag{2.14}$$

The origin of the hole interpretation is thus the crossing of particles from bra to ket by means of (2.6) together with the renormalization prescription. As explained in detail in [6], in comparison with the one-point function there are additional singular terms emerging in the two-point functions. In the expansion they occur in terms in which the product of the two form factors associated to $\mathcal{O}$ and $\mathcal{O}'$ involve a different amount of particles. These terms are just dropped, which could be a possible source for the difficulties we encounter below in interacting theories.

The functions $f_i(\theta,T)$ are the so-called filling fractions

$$f_i(\theta,T)=\frac{1}{1-S_{ii}(0)\exp[-\varepsilon_i(\theta,T)]} \tag{2.15}$$

involving the functions $\varepsilon_i(\theta,T)$. For the noninteracting case, treated in [6], this function is taken to be the on-shell energy divided by the temperature $\varepsilon_i(\theta,T)=m_i/T\cosh\theta$. When extending the validity of these formulae to the interacting case, one may speculate on the nature of $\varepsilon_i(\theta,T)$. In [7] it was proposed to interpret $\varepsilon_i(\theta,T)$ as the pseudo-energies, which may be determined by means of the thermodynamic Bethe ansatz equation [15]

$$\varepsilon_i(\theta,\hat{r}) = \hat{r}\, m_i \cosh\theta + \sum\nolimits_j [\varphi_{ij} * \ln f_j](\theta). \tag{2.16}$$

As common we denote here $\hat{r} = m/T$, $m_l \to m_l/m$, with m being the mass of the lightest particle in the model. By $(f * g)(\theta) := 1/(2\pi)\int d\theta'\, f(\theta-\theta')g(\theta')$ we denote the convolution of two functions and $\varphi_{ij}(\theta) = -i\, d\ln S_{ij}(\theta)/d\theta$. In contrast, when specializing the statement in [7] to fermionic statistics, i.e., $S_{ii}(0) = -1$, it was suggested therein to take instead the free on-shell energies divided by the temperature also in the interacting case. More generally, this means that the eigenvalue E_ψ of the Hamiltonian in (2.4) is either taken to be the free on-shell or the pseudo-energy. In each case, the pseudo-energies of the holes are simply the negative of the ones of the particles

$$\varepsilon_{\text{hole}}(\theta,T) = -\varepsilon_{\text{particle}}(\theta,T). \tag{2.17}$$

A simple but key property satisfied by the two-point function is the Kubo–Martin–Schwinger (KMS)-condition [16]. Assuming to have a time evolution operator ω_t at disposal, it is easily obtained from the trace properties of the temperature dependent correlation function

$$\begin{aligned}\langle \omega_t \mathcal{O}\mathcal{O}'\rangle_T = \langle \mathcal{O}'\omega_{t+i\beta}\mathcal{O}\rangle_T \Longleftrightarrow \langle \mathcal{O}(x,t)\mathcal{O}'(x',t')\rangle_T \\ = \langle \mathcal{O}'(x',t')\mathcal{O}(x,t+i\beta)\rangle_T.\end{aligned} \tag{2.18}$$

For more detailed discussion on this formula see, e.g., [17]. For the choice $x^\mu = (-ir, 0)$, as in (2.13), this condition reads

$$\langle \omega_t \mathcal{O}\mathcal{O}'\rangle_T = \langle \mathcal{O}'\omega_{t-\beta}\mathcal{O}\rangle_T \quad\Leftrightarrow\quad \langle \mathcal{O}(r)\mathcal{O}'(0)\rangle_T = \langle \mathcal{O}'(0)\mathcal{O}(r-\beta)\rangle_T. \tag{2.19}$$

Noting that $\langle \mathcal{O}(r)\mathcal{O}'(0)\rangle_T = \langle \mathcal{O}(0)\mathcal{O}'(-r)\rangle_T$, it is clear that (2.13) indeed satisfies the condition (2.19), provided that the form factors obey

$$\begin{aligned}&F_n^{\mathcal{O}|n\times\text{ holes } m\times\text{ particles}}(\theta_1,\ldots,\theta_{n+m})\big[F_n^{\mathcal{O}'|n\times\text{ holes } m\times\text{ particles}}(\theta_1,\ldots,\theta_{n+m})\big]^* \\ &\quad = F_n^{\mathcal{O}|n\times\text{ particles } m\times\text{ holes}}(\theta_1,\ldots,\theta_{n+m}) \\ &\qquad \times \big[F_n^{\mathcal{O}'|n\times\text{ particles } m\times\text{ holes}}(\theta_1,\ldots,\theta_{n+m})\big]^*.\end{aligned} \tag{2.20}$$

However, from one of Watson's equations (2.9) and Lorentz invariance (2.10) we easily derive that in general one picks up a factor $\exp i\pi(s'-s)$ on the r.h.s. of (2.20), where s, s' are the Lorentz spins of $\mathcal{O}$ and $\mathcal{O}'$, respectively. Consequently, there is no problem with KMS when $(s'-s) \in 2\mathbb{Z}$ like for instance when the two operators coincide. The latter case was treated in [6–10]. Despite the fact that the KMS condition puts some structural restrictions on the two-point functions, it is not constraining enough in what more precise functional details concern, e.g., it cannot shed any light on the controversy [7,9,10] about the precise nature of the dressing function (2.15). However, it dictates the two functions $\varepsilon_\mu(\theta,T)$ appearing explicitly in (2.13) and (2.15) to be identical.

To clarify further the working of (2.13) it would be highly desirable to compute this functions for more explicit models. Unfortunately there are not many temperature dependent two-point functions known from alternative approaches which one could compare with in order to settle the issue. Nonetheless, various limits are known which one can take as benchmarks.

2.1. The massless limit, conformal correlation functions

In conformal field theory various methods have been developed to compute correlation functions. At vanishing temperature the most celebrated approach is the one which exploits the structure of the Virasoro algebra, such that the correlation functions obey certain differential equations [18]. To include the temperature is also fairly simple in this case. It is achieved just by mapping the observables from the plane to the cylinder, $z \to \exp(2\pi T\vartheta)$, $\mathcal{O}(z) \to (2\pi T)^{-\Delta_{\mathcal{O}}} e^{-2\pi T\vartheta\Delta_{\mathcal{O}}} \mathcal{O}(\vartheta)$

$$\begin{aligned}&\big\langle \mathcal{O}(\vartheta_1,\bar\vartheta_1)\mathcal{O}'(\vartheta_2,\bar\vartheta_2)\big\rangle_T \\ &\quad = \big\langle \mathcal{O}(\vartheta_1)\mathcal{O}'(\vartheta_2)\big\rangle_T \big\langle \mathcal{O}(\bar\vartheta_1)\mathcal{O}'(\bar\vartheta_2)\big\rangle_T \\ &\quad = (2\pi T)^{\Delta_{\mathcal{O}}+\Delta_{\mathcal{O}'}+\bar\Delta_{\mathcal{O}}+\bar\Delta_{\mathcal{O}'}} \big\langle \mathcal{O}(z_1)\mathcal{O}'(z_2)\big\rangle_{T=0} \big\langle \mathcal{O}(\bar z_1)\mathcal{O}'(\bar z_2)\big\rangle_{T=0}.\end{aligned} \tag{2.21}$$

By construction (2.21) satisfies the KMS condition, provided

$$\big\langle \mathcal{O}(z_1)\mathcal{O}'(z_2)\big\rangle_{T=0} = \big\langle \mathcal{O}'(z_2)\mathcal{O}(z_1)\big\rangle_{T=0}.$$

Alternatively one can get some further information on this function by exploiting the KMS condition on one of the holomorphic sectors by adopting a proposal made in [20]. Ignoring normal ordering one obtains

$$\big\langle \mathcal{O}(\vartheta_1)\mathcal{O}'(\vartheta_2)\big\rangle_T = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \exp\big[-2\pi i T(\vartheta_1 n + \vartheta_2 k)\big]\big[\mathcal{O}_n, \mathcal{O}'_k\big]. \tag{2.22}$$

For this to hold we only need to assume that for $\mathcal{O}(\vartheta_1)$ and $\mathcal{O}'(\vartheta_2)$ exist Fourier–Laurent mode expansions of the form

$$\mathcal{O}(\vartheta) = \sum_{n=-\infty}^{\infty} \exp(-2\pi n i T\vartheta)\mathcal{O}_n. \tag{2.23}$$

In addition one makes use of the fact that the time evolution is governed by $\omega_t\mathcal{O} = e^{itL_0}\mathcal{O}e^{-itL_0}$, with L_0 being the zero mode generator of the Virasoro algebra. If then furthermore the commutator $[L_0, \mathcal{O}_n] = -n\mathcal{O}_n$ holds (this is true for instance for $\mathcal{O}_n = L_n$ the modes of the energy–momentum tensor, $\mathcal{O}_n = J_n^a$ the modes of a Kac–Moody current, $\mathcal{O}_n = \phi_n$ the modes of a primary field, $\mathcal{O}_n = G_n$ the modes of an $N = 1$ supersymmetric field),[1] the relation (2.22) is derived immediately. To make contact with (2.21) one needs of course to incorporate a proper normal ordering prescription.

Alternatively, we may compute the correlation functions by using the (dressed) form factors related to the massless theory. The prescription of taking the massless limit was originally introduced in [21] within the context of a scattering theory. It consists of replacing in every rapidity dependent expression $\theta \to \theta \pm \sigma$, where an additional auxiliary parameter σ has been introduced. Thereafter one should take the limit $\sigma \to \infty$, $m \to 0$ while keeping the quantity $\hat m = m/2\exp(\sigma)$ finite. For instance, carrying out this

[1] In [19] it was shown, however, that supersymmetry and temperature seem to be incompatible concepts. Only the vacuum states admit the implementation of supersymmetry.

prescription for the momentum yields $p_\pm = \pm\hat{m}\exp(\pm\theta)$, such that one may view the model as splitted into its two chiral sectors and one can speak naturally of left (L) and right (R) movers. Hence, having a function depending on the rapidities of n particles, it will be mapped into 2^n related functions

$$\lim_{\sigma\to\infty} f(\theta_1+\varkappa_1\sigma,\ldots,\theta_n+\varkappa_n\sigma) \\ = f_{\nu_1,\ldots,\nu_n}(\theta_1,\ldots,\theta_n), \quad \begin{cases} \nu_i = L & \text{for } \varkappa_i = -, \\ \nu_i = R & \text{for } \varkappa_i = +. \end{cases} \tag{2.24}$$

For the scattering matrix this means every massive amplitude is duplicated $S(\theta) = S_{LL}(\theta) = S_{RR}(\theta)$ and in addition one obtains the amplitudes connecting the two chiral sectors $S_{RL/LR}(\theta) = \lim_{\sigma\to\infty} S(\theta \pm 2\sigma)$.

In [3] this prescription was also carried out for expressions of form factors. In that case each n-particle form factor is turned into 2^n n-particle form factors. Note that when considering (2.24) for form factors this does in general not lead to the same expressions as when taking the scattering matrices $S_{LL}(\theta), S_{RR}(\theta), S_{RL}(\theta), S_{LR}(\theta)$ and determining the form factors thereafter in the usual fashion. This two ways of carrying out the limit only commute for form factors associated to operators whose Lorentz spin is vanishing.

In order to be able to formulate the analogue of the expression (2.13) for the massless case one also requires the massless version of the dressing function, i.e., the massless analogues of the pseudo-energies. For this one can use once more the above recipe, such that the TBA-equations (2.16) are replaced by the same equations with $S(\theta) \to S_{LL}(\theta), S_{RR}(\theta), S_{RL}(\theta), S_{LR}(\theta)$ and $\hat{r}\, m_i \cosh\theta \to \hat{r}\, \hat{m}_i \exp\theta$. The working of this was confirmed in the analysis of [21].

Having now outlined the prescription to compute the massless (temperature dependent) correlation functions, a nontrivial check is constituted by the comparison with (2.21). A more constricting check is to start with (2.13) and carry out the massless and zero temperature limit. That is checking the commutativity of the diagram in Fig. 1. In

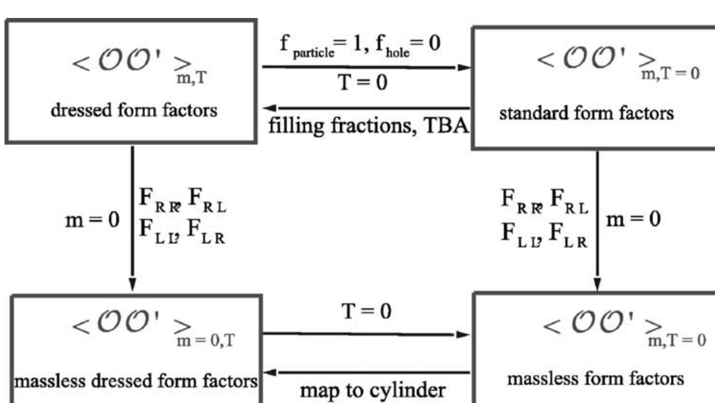

Fig. 1. Two-point correlation functions in various limits.

particular, the massless limit can possibly shed more light on the issue of different dressing functions.

3. The complex free Fermion/Federbush model

We will now demonstrate the working of the previous approach with various examples. Let us start with the complex free Fermion case (the complex version of the Ising model) with Virasoro central charge $c = 2 \times 1/2$. One reason for not considering directly the simpler case of self-conjugate Fermions is that most of the formulae presented in this section also hold for the more general Federbush model [24,25], which can be viewed as an anyonic generalization of the complex free Fermion. This means we can also regard the results of this section to hold for more exotic statistical interaction, since apart from the disorder field the operators are formally identical. In general, the free Fermion is particularly attractive to start with, since for many operators the higher n-particle form factors are vanishing such that the infinite series in (2.13) terminates. For the self-conjugate Fermion several two-point correlation functions at zero temperature have been computed, e.g., in [22,23]. Despite the simplicity of the model, only for very few operators massive temperature dependent correlation functions have been evaluated by means of form factors [6]. In order to put formula (2.13) on firmer ground it is therefore desirable to check its working first of all for a wider range of operators.

Taking the operator $\mathcal{O}$ and $\mathcal{O}'$ to be the current, we will present all four cases illustrated in Fig. 1 in some detail. The current–current correlation functions are particularly interesting, since they occur explicitly in the application within the Kubo formula. Adopting the notation of our recent exposition [25], we consider the correlation function involving one of the chiral currents $J^{\pm} = J^0 \pm J^1$. The only nonvanishing massive form factors for these operators are

$$F_2^{J^{\pm}|\bar{\imath}i}(\theta, \tilde{\theta}) = -F_2^{J^{\pm}|i\bar{\imath}}(\theta, \tilde{\theta}) = -i\pi m e^{\mp\frac{\theta+\tilde{\theta}}{2}}. \tag{3.1}$$

In the following we shall focus on the mutual correlator of $J \equiv J^-$. Out of the four cases depicted in Fig. 1, the conformal case at zero temperature is the easiest to treat and hence a good starting point. According to the massless limit prescription (2.24), we obtain

$$F_{2,RR}^{J|\bar{\imath}i}(\theta, \tilde{\theta}) = -F_{2,RR}^{J|i\bar{\imath}}(\theta, \tilde{\theta}) = -2\pi i\, \hat{m} \exp(\theta + \tilde{\theta})/2, \tag{3.2}$$

$$F_{2,LL}^{J|\bar{\imath}i}(\theta, \tilde{\theta}) = F_{2,LR}^{J|\bar{\imath}i}(\theta, \tilde{\theta}) = F_{2,RL}^{J|\bar{\imath}i}(\theta, \tilde{\theta}) = 0, \tag{3.3}$$

$$F_{2,LL}^{J|i\bar{\imath}}(\theta, \tilde{\theta}) = F_{2,LR}^{J|i\bar{\imath}}(\theta, \tilde{\theta}) = F_{2,RL}^{J|i\bar{\imath}}(\theta, \tilde{\theta}) = 0, \tag{3.4}$$

such that (2.13) yields

$$\begin{aligned}&\langle J(r) J(0)\rangle_{m=0,T=0}\\&\quad = 4\hat{m}^2\pi^2 \int_{-\infty}^{\infty} \frac{d\theta\, d\tilde{\theta}}{(2\pi)^2} \exp\bigl[-r\,\hat{m}\bigl(e^{\theta} + e^{\tilde{\theta}}\bigr)\bigr]\bigl(e^{\theta+\tilde{\theta}}\bigr) = \frac{1}{r^2}.\end{aligned} \tag{3.5}$$

Recalling that the current has scaling dimension $\Delta_J = 1$, $\bar{\Delta}_J = 0$, this agrees of course with the leading order term of the well-known conformal $U(1)$-current–current two-point correlation function k/r^2 for level $k = 1$, see, e.g., [26]. Note, that as it should be, the auxiliary parameter $\hat{m}$ has vanished in the final expressions. Raising now the temperature we can use the same expressions for the form factors, but the proposal (2.13) dictates that we have to dress them with the massless version of the filling fractions

$$\hat{f}_\pm(\theta, T) = \frac{1}{1 + \exp(\mp\hat{m}/T e^\theta)} \tag{3.6}$$

for particles (+) and holes (−). The values are in agreement with (2.17). With this we compute

$$\begin{aligned}
&\langle J(r)J(0)\rangle_{m=0,T} \\
&\quad = 4\hat{m}^2\pi^2 \sum_{\mu,\nu=\pm} \int_{-\infty}^{\infty} \frac{d\theta\, d\tilde{\theta}}{(2\pi)^2} \big(e^{\theta+\tilde{\theta}}\big) \hat{f}_\mu(\theta, T) \hat{f}_\nu\big(\tilde{\theta}, T\big) e^{-r\hat{m}(\mu e^\theta + \nu e^{\tilde{\theta}})} \\
&\quad = \frac{\pi^2 T^2}{\sin^2(\pi r T)}.
\end{aligned} \tag{3.7}$$

The result (3.7) can of course also be obtained directly from the mapping (2.21) and the correlation function at zero temperature (3.5). Making now the model massive, we employ instead of (3.2) and (3.3) the form factors (3.1) and evaluate

$$\begin{aligned}
&\langle J(r)J(0)\rangle_{m,T=0} \\
&\quad = m^2\pi^2 \int_{-\infty}^{\infty} \frac{d\theta\, d\tilde{\theta}}{(2\pi)^2} \exp\big[-rm\big(\cosh\theta + \cosh\tilde{\theta}\big)\big]\big(e^{\theta+\tilde{\theta}}\big) \\
&\quad = m^2\big[K_1(rm)\big]^2,
\end{aligned} \tag{3.8}$$

where $K_1(x)$ is a modified Bessel function (see Appendix A). Using the limiting behaviour (A.2), we recover as expected the conformal correlation function (3.5) in the limit $m \to 0$. Considering now the massive finite temperature regime, we have to include in the previous computation the massive dressing function

$$f_\pm(\theta, T) = \frac{1}{1 + \exp(\mp m/T \cosh\theta)}. \tag{3.9}$$

Then we compute according to (2.13)

$$\begin{aligned}
&\langle J(r)J(0)\rangle_{m,T} \\
&\quad = m^2\pi^2 \sum_{\mu,\nu=\pm} \int_{-\infty}^{\infty} \frac{d\theta\, d\tilde{\theta}}{(2\pi)^2} \big(e^{\theta+\tilde{\theta}}\big) f_\mu(\theta, T) f_\nu\big(\tilde{\theta}, T\big) e^{-rm(\mu\cosh\theta + \nu\cosh\tilde{\theta})} \\
&\quad = m^2\big[\widehat{K}_1^+(m, r, T)\big]^2.
\end{aligned} \tag{3.10}$$

The functions $\widehat{K}_{\alpha}^{\pm}(m,r,T)$ are defined in Appendix A. The restriction on the arguments of the Bessel functions (A.1) reflects the conditions $r < 1/T$ in (2.13). With the help of (A.10) we recover the expression (3.8) for $T \to 0$. Taking instead first the limit $m \to 0$ in (3.10), we reproduce with (A.5) the previously computed conformal correlator (3.7). In conclusion this means the different methods to compute $\langle J(r)J(0)\rangle$ for several mass and temperature regimes are consistent and indeed the diagram in Fig. 1 is commutative for the considered choice of operators.

We proceed now similarly and compute the two-point correlation functions involving various other operators. In what follows we will be less explicit as for $\langle J(r)J(0)\rangle$ in the derivation of the finite temperature and mass correlation function and just quote the final results. Thereafter, we carry out the various limits by using the formulae quoted in Appendix A.

Recalling that the only nonzero form factors of the trace of the energy–momentum tensor Θ are

$$F_2^{\Theta|\bar{\imath}i}(\theta,\tilde{\theta}) = F_2^{\Theta|i\bar{\imath}}(\theta,\tilde{\theta}) = -2\pi i m^2 \sinh(\theta-\tilde{\theta})/2, \tag{3.11}$$

we obtain, according to (2.13), for its mutual correlation function

$$\langle \Theta(r)\Theta(0)\rangle_{m,T} = 2m^4\left[\widehat{K}_1^{+}(m,r,T)^2 - \widehat{K}_0^{-}(m,r,T)^2\right]. \tag{3.12}$$

We can verify the commutativity of the diagram in Fig. 1 similarly as for the current–current correlator

$$\lim_{m\to 0}\left[\lim_{T\to 0}\langle \Theta(r)\Theta(0)\rangle_{m,T}\right] = \lim_{m\to 0}\left[2m^4\left(K_1^2(rm) - K_0^2(rm)\right)\right] = 0, \tag{3.13}$$

$$\lim_{T\to 0}\left[\lim_{m\to 0}\langle \Theta(r)\Theta(0)\rangle_{m,T}\right] = \lim_{T\to 0}[0] = 0. \tag{3.14}$$

The conformal limits in (3.13) and (3.14) reflect of course the vanishing of the trace of the energy–momentum tensor. Noting that the energy density operator ϵ with conformal dimension $\Delta_\epsilon = \bar{\Delta}_\epsilon = 1/2$ is related to the trace as $\Theta = m\epsilon$, we obtain more interesting limits

$$\lim_{m\to 0}\left[\lim_{T\to 0}\langle \epsilon(r)\epsilon(0)\rangle_{m,T}\right] = \lim_{m\to 0}\left[2m^2\left(K_1^2(rm) - K_0^2(rm)\right)\right] = \frac{2}{r^2}, \tag{3.15}$$

$$\lim_{T\to 0}\left[\lim_{m\to 0}\langle \epsilon(r)\epsilon(0)\rangle_{m,T}\right] = \lim_{T\to 0}\left[\frac{2\pi^2T^2}{\sin^2(\pi r T)}\right] = \frac{2}{r^{2\Delta_\epsilon+2\bar{\Delta}_\epsilon}}, \tag{3.16}$$

which are again consistent. Recalling that for the $(++)$-component of the energy–momentum tensor $T^{++} \equiv \overline{T}$ with $\Delta_{\overline{T}} = 2$, $\bar{\Delta}_{\overline{T}} = 0$ the only nonvanishing form factors are

$$F_2^{\overline{T}|\bar{\imath}i}(\theta,\tilde{\theta}) = F_2^{\overline{T}|i\bar{\imath}}(\theta,\tilde{\theta}) = \pi i/2 m^2 \exp(\theta+\tilde{\theta})\sinh(\theta-\tilde{\theta})/2. \tag{3.17}$$

The mutual correlation function results to

$$\langle \overline{T}(r)\overline{T}(0)\rangle_{m,T} = \frac{m^4}{8}\left[\widehat{K}_3^{+}(m,r,T)\widehat{K}_1^{+}(m,r,T) - \widehat{K}_2^{-}(m,r,T)^2\right]. \tag{3.18}$$

Once again the commutativity of the diagram in Fig. 1 is confirmed

$$\lim_{m\to 0}\Big[\lim_{T\to 0}\big\langle \overline{T}(r)\overline{T}(0)\big\rangle_{m,T}\Big] = \lim_{m\to 0}\left[\frac{m^4}{8}\big(K_1(rm)K_3(rm) - K_2^2(rm)\big)\right] = \frac{1}{2r^4}, \tag{3.19}$$

$$\lim_{T\to 0}\Big[\lim_{m\to 0}\big\langle \overline{T}(r)\overline{T}(0)\big\rangle_{m,T}\Big] = \lim_{T\to 0}\left[\frac{1}{2}\frac{\pi^4 T^4}{\sin^4(\pi r T)}\right] = \frac{c}{2r^{2\Delta_{\overline{T}}+2\bar{\Delta}_{\overline{T}}}}. \tag{3.20}$$

We also compute

$$\big\langle \overline{T}(r)\Theta(0)\big\rangle_{m,T} = \frac{m^4}{2}\big[\widehat{K}_1^-(m,r,T)^2 - \widehat{K}_0^+(m,r,T)\widehat{K}_2^+(m,r,T)\big], \tag{3.21}$$

together with the expected limiting behaviour

$$\lim_{m\to 0}\Big[\lim_{T\to 0}\big\langle \overline{T}(r)\Theta(0)\big\rangle_{m,T}\Big] = \lim_{m\to 0}\left[\frac{m^4}{2}\big(K_1^2(rm) - K_0(rm)K_2(rm)\big)\right] = 0, \tag{3.22}$$

$$\lim_{T\to 0}\Big[\lim_{m\to 0}\big\langle \overline{T}(r)\Theta(0)\big\rangle_{m,T}\Big] = \lim_{T\to 0}[0] = 0. \tag{3.23}$$

Once again, the conformal limits in (3.22) and (3.23) reflect the vanishing of the trace of the energy–momentum tensor. Replacing $\Theta \to m\epsilon$ will only change in (3.22) $m^4 \to m^3$ and the remaining limits are the expected ones.

However, there are also operators for which the prescription does nor work so smoothly. As an example, we now want to compute $\langle \overline{T}(r)\mu(0)\rangle_{m,T}$, with μ being the disorder field of the complex free Fermion theory. For this purpose, it is necessary first to recall the expressions of the form factors related to this field, which were computed in [25], and shown to be different from the ones associated to the counterpart of this field in the Ising model. Similarly as for the latter model, it was found that only the form factors involving an even number of particles are nonzero. Since for $\overline{T}$ the only nonvanishing form factors are the two-particle ones, the only nonvanishing form factors of the field μ which will be relevant for these computations are

$$F_2^{\mu|\bar{\imath}i}\big(\theta,\tilde{\theta}\big) = -F_2^{\mu|i\bar{\imath}}\big(-\theta,-\tilde{\theta}\big) = i/2\langle\mu\rangle_{T=0}\exp\big[\big(\theta-\tilde{\theta}\big)/2\big]\cosh^{-1}\big(\theta-\tilde{\theta}\big)/2. \tag{3.24}$$

With these data we compute, again according to (2.13),

$$\begin{aligned}\big\langle \overline{T}(r)\mu(0)\big\rangle_{m,T} &= \frac{T^3\langle\mu\rangle_{T=0}}{16}\int_r^{1/T} dt\, e^{-2tm}\left[\frac{2m}{T}\Phi\left(-e^{-\frac{m}{T}},\frac{1}{2},tT\right)\Phi\left(-e^{-\frac{m}{T}},\frac{3}{2},tT\right)\right.\\ &\quad - \Phi\left(-e^{-\frac{m}{T}},\frac{3}{2},tT\right)^2 + 3\Phi\left(-e^{-\frac{m}{T}},\frac{1}{2},tT\right)\Phi\left(-e^{-\frac{m}{T}},\frac{5}{2},tT\right)\\ &\quad \left. - t\to\frac{1}{T}-t\right].\end{aligned} \tag{3.25}$$

We employed Lerch's transcendental function $\Phi(x,s,\alpha)$ (see Appendix A). In comparison with our previous computations we have one integration remaining. This results from the fact that unlike before we have now a term $(\cosh\tilde{\theta}+\cosh\theta)$ in the denominator, which we eliminate by a differentiation with respect to r and a subsequent integration. Alternatively, we obtain the same result by a direct computation using a variable substitution similar as in [22] $\sinh\theta/2 = r\cos\phi$, $\sinh\tilde{\theta}/2 = r\sin\phi$. By employing (A.14) the limits come out as

$$\lim_{m\to 0}\Big[\lim_{T\to 0}\big\langle\overline{T}(r)\mu(0)\big\rangle_{m,T}\Big] = \lim_{m\to 0}\left[\langle\mu\rangle_{T=0}\frac{e^{-2rm}}{16r^2}\right] = \frac{\langle\mu\rangle_{T=0}}{16r^2}. \tag{3.26}$$

However, starting with the limit $m\to 0$ in (3.25) is problematic, because the expression $\lim_{x\to -1}\Phi(x,s,\alpha)$ for $\operatorname{Re}s<1$ is not well defined. We obtain similar phenomena for other correlation functions involving different operators, such as $\langle J(r)\mu(0)\rangle_{T,m}$.

In principle one could extend this list of correlation functions involving various other operators and support more and more our overall conclusion, namely, that the conjecture of a dressed form factor expansion (2.13) is meaningful for free theories even when the underlying statistics is anyonic. Next we want to see whether the picture still remains the same for dynamically interacting theories.

4. The scaling Yang–Lee model

The scaling Yang–Lee model (or minimal $A_2^{(2)}$-affine Toda field theory), like its conformal counterpart with Virasoro central charge $c=-22/5$, is one of the simplest interacting integrable quantum field theories in $1+1$ space–time dimensions. It is an ideal starting point to test general ideas, since it is comprised of only one massive particle which couples to itself. This is reflected by the pole in the physical sheet of its scattering matrix

$$S_{\mathrm{YL}}(\theta) = \frac{\sinh\theta + i\sin\pi/3}{\sinh\theta - i\sin\pi/3} \tag{4.1}$$

which was proposed in [27]. Closed formulae for all n-particle form factors for various components of the energy–momentum tensor were computed in [28]. For our purposes we will just require the ones up to two particles. We recall from [28] in a slightly different notation the form factors associated to $T^{++}\equiv\overline{T}$

$$F_0^{\overline{T}} = -\frac{\pi m^2}{\sqrt{3}}, \qquad F_1^{\overline{T}}(\theta) = -\frac{i\pi m^2}{\nu 2^{5/2}3^{1/4}}e^{2\theta}, \tag{4.2}$$

$$F_2^{\overline{T}}(\theta_1,\theta_2) = \frac{\pi m^2}{8}\frac{\cosh\theta_{12}-1}{\cosh\theta_{12}+1/2}e^{\theta_1+\theta_2}F_{\min}(\theta_{12}), \tag{4.3}$$

where the so-called minimal form factor is

$$F_{\min}(\theta) = \exp\left[-8\int_0^\infty\frac{dt}{t}\frac{\sinh\frac{t}{2}\sinh\frac{t}{3}\sinh\frac{t}{6}}{\sinh^2 t}\sin^2\left(\frac{t(i\pi-\theta)}{2\pi}\right)\right] \tag{4.4}$$

and ν is a constant given by

$$\nu = \exp\left(2\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{t}{2}\sinh\frac{t}{3}\sinh\frac{t}{6}}{\sinh^2 t}\right) = 1.11154\ldots. \tag{4.5}$$

Our aim is to use these expressions and compute by means of the dressed form factor formula (2.13) the two-point correlation functions. Unfortunately, to our knowledge there exists no computation in the massive and temperature dependent situation to compare with. However, in the massless case we have two benchmarks, namely

$$\langle \overline{T}(r)\overline{T}(0)\rangle_{m=T=0} = -\frac{11}{5}\frac{1}{r^4}, \tag{4.6}$$

$$\langle \overline{T}(r)\overline{T}(0)\rangle_{m=0,T} = -\frac{11}{5}\frac{\pi^4 T^4}{\sin^4(\pi r T)}. \tag{4.7}$$

Here (4.6) is just the well known two-point function from conformal field theory $\langle \overline{T}(r)\overline{T}(0)\rangle_{m=T=0} = c/2r^{-4}$, with $c = -22/5$, and (4.7) is this formula mapped to the cylinder according to (2.21) with $\Delta_{\overline{T}} = 2$. Let us therefore compute the massless scattering matrices and form factors from (4.1) and (4.2), (4.3), respectively. According to the prescription (2.24) outlined at the end of Section 2, we compute

$$S_{RR}(\theta) = S_{LL}(\theta) = S_{\mathrm{YL}}(\theta), \qquad S_{RL}(\theta) = S_{LR}(\theta) = 1 \tag{4.8}$$

and

$$F_0^{\overline{T}} = F_{1,L}^{\overline{T}} = F_{2,LL}^{\overline{T}} = F_{2,LR}^{\overline{T}} = F_{2,RL}^{\overline{T}} = 0, \tag{4.9}$$

$$F_{1,R}^{\overline{T}}(\theta) = -\frac{i\pi\,\hat{m}^2}{\nu 2^{1/2}3^{1/4}}e^{2\theta}, \tag{4.10}$$

$$F_{2,RR}^{\overline{T}}(\theta_1,\theta_2) = \frac{\pi\,\hat{m}^2}{2}\frac{\cosh\theta_{12} - 1}{\cosh\theta_{12} + 1/2}e^{\theta_1+\theta_2}F_{\min}(\theta_{12}). \tag{4.11}$$

These expressions also exemplify our remark at the end of Section 2, namely, that one cannot take the scattering matrices (4.8) and compute the form factors thereafter from first principles. In that case we would obtain an L and LL contribution, which are evidently vanishing when we carry out the method directly on the formulae (4.2) and (4.3) for the massive regime. The reason is simply that the "massless prescription" is only a way to carry out the limit starting with the massive expressions, but not a first principle concept. Nonetheless, viewing it in this sense it works extremely well.

Let us commence with the zero temperature case. The one-particle contribution can be computed analytically

$$\begin{aligned}\langle \overline{T}(r)\overline{T}(0)\rangle^{(1)}_{m=T=0} &= -\int \frac{d\theta}{2\pi}\left|F_{1,R}^{\overline{T}}(\theta)\right|^2 e^{-\hat{m}re^{\theta}} \\ &= -\frac{\pi\sqrt{3}}{2\nu^2}\frac{1}{r^4} = -\frac{2.2020498\ldots}{r^4}.\end{aligned} \tag{4.12}$$

Table 1
Two-particle contribution to $\langle \overline{T}(r)\overline{T}(0)\rangle_{m=T=0}$

r	0.0001	0.0003	0.0005	0.001	0.01	0.1
$r^4\langle \overline{T}(r)\overline{T}(0)\rangle^{(2)}_{m=T=0}10^3$	0.5631	1.9435	2.0457	2.0487	2.0487	2.0487

We observe, that the expected value is already almost saturated. To compute the two particle contribution is a fairly simple numerical exercise. We evaluate

$$\langle \overline{T}(r)\overline{T}(0)\rangle^{(2)}_{m=T=0} = \int \frac{d\theta_1\, d\theta_2}{2(2\pi)^2} \left| F^{\overline{T}}_{2,RR}(\theta_1,\theta_2)\right|^2 e^{-\hat{m}r(e^{\theta_1}+e^{\theta_2})}, \tag{4.13}$$

and present our results in Table 1

As mentioned in [28], when summing up according to (2.13) with $f_\mu(\theta, T) = 1$, this contribution enters with a positive sign in comparison with the one-particle contribution due to the nonunitarity of the model. As expected, similar to the ($T = 0$, $m \neq 0$)-case, carried out in [28], we observe an extremely fast convergence of the series towards the expected value (4.6). The numbers in Table 1 confirm the general observation, which was also made in [28], that for extremely small values of r the higher particle contributions become more important. Considering a regime for $r > 0.001$, it will not be necessary to include also the three-particle contribution in order to reach our main conclusion. Nonetheless, in principle this could be done easily with some Monte Carlo integration, just at the cost of longer computing time, since in [28] all n-particle form factors were already provided.

Notice further that the parameter $\hat{m}$ plays no role anymore. As observed before, in the analytical computation (4.12) it cancels explicitly. This phenomenon is less apparent in the two-particle formula (4.13), but we convinced ourselves that $\hat{m}$ may be re-scaled without altering the outcome of the numerical computation. Thus the expressions are mass independent as they should be. In what follows we will use this fact to simplify notations and scale $\hat{m}$ to one.

Let us now embark upon the nonzero temperature case. By means of the conjecture (2.13), we have to compute for the one-particle contribution

$$\begin{aligned}&\langle \overline{T}(r)\overline{T}(0)\rangle^{(1)}_{m=0,T}\\ &\quad = -\int \frac{d\theta}{2\pi}\Big(f_+(\theta,T)\left|F^{\overline{T}}_{1,R}(\theta)\right|^2 e^{-rT\varepsilon(\theta,T)}\\ &\qquad\qquad + f_-(\theta,T)\left|F^{\overline{T}\overline{T}}_{1,R}(\theta - i\pi)\right|^2 e^{rT\varepsilon(\theta,T)}\Big) e^{4\theta}.\end{aligned} \tag{4.14}$$

According to the conjectures in [7,10] or if we extend the proposal in [9] to the two-point function, we can choose for the $\varepsilon(\theta,T)$ in (4.14) and the corresponding dressing functions (2.15) either $\varepsilon_{\mathrm{TBA}}(\theta,T)$, determined by the massless version of the TBA-equation (2.16) or $\varepsilon_{\mathrm{free}}(\theta,T) = e^\theta/T$, respectively. Solving first numerically the massless TBA-equation by means of a standard iteration procedure, we can compare the two dressing functions $f_\pm(\varepsilon_{\mathrm{TBA}}(\theta,T))$ and $f_\pm(\varepsilon_{\mathrm{free}}(\theta,T))$. Our results are depicted in Fig. 2.

We plotted the corresponding functions for the left and right movers in order to exhibit the symmetry of the TBA solutions. The solutions for the left movers are not important

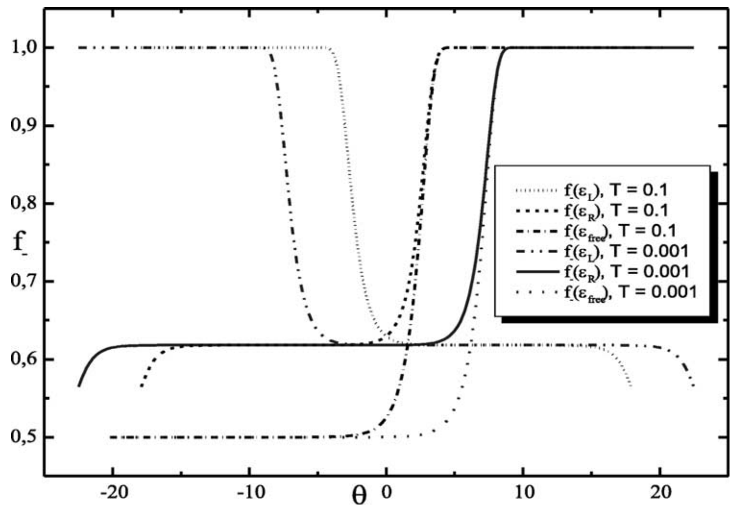

Fig. 2. Two possible dressing functions $f_\pm(\varepsilon_{\rm TBA}(\theta,T))$ and $f_\pm(\varepsilon_{\rm free}(\theta,T))$.

for what follows. We observe that $f_-(\varepsilon_{\rm TBA}(\theta,T))$ acquires plateaux at $f_-(\varepsilon_{\rm TBA}(\theta,T)) = 2/(1+\sqrt{5})$ which are characteristic for all minimal affine Toda field theories. Furthermore, we see the well-known fact that in the large rapidity regime the TBA solutions merge with the free one. Nonetheless, the two functions $f_\pm^{\rm TBA}(\theta,T)$ and $f_\pm^{\rm free}(\theta,T)$ differ quite substantially within a large rapidity regime. However, in the computation in which they are actually needed, namely in (4.14), this regime is negligible. This is essentially due to the factor $e^{4\theta}$. This means the issue of controversy on the difference between the two functions $f_\pm^{\rm TBA}(\theta,T)$ and $f_\pm^{\rm free}(\theta,T)$ is irrelevant if we would extend it to the context of the two-point functions.

Let us therefore take $f_\pm^{\rm free}(\theta,T)$, for which we can compute (4.14) analytically

$$\left\langle \overline{T}(r)\overline{T}(0)\right\rangle_{m=0,T}^{(1)} = -\frac{\pi\sqrt{3}}{2\nu^2}T^4\left[\Phi(-1,4,rT)+\Phi(-1,4,1-rT)\right]. \tag{4.15}$$

Here $\Phi(x,s,\alpha)$ is again Lerch's transcendental function, which was already encountered in Section 3 (see also Appendix A). Extrapolating the behaviour from the cases ($m \neq 0$, $T=0$) and ($T=0, m=0$), we expect that the main contribution to the sum comes from the one-particle form factors. It is therefore instructive to compare the ratio of the expected value (4.7) and $\langle \overline{T}(r)\overline{T}(0)\rangle_{m=0,T\neq 0}^{(1)}$. We depict this comparison in Fig. 3.

We observe a deviation of up to 300%, which when recalling the excellent agreement at this level of the cases ($m \neq 0$, $T=0$) and ($m=0$, $T=0$), sheds a rather pessimistic light on the working of the conjecture (2.13) in the interacting case. In fact, we only observe a reasonable match when $0 < rT < 0.01$ or $0.99 < rT < 1.0$, but this is unfortunately just a regime in which we can approximate in (4.7) $\sin^4(\pi rT) \sim (\pi rT)^4$ such that the temperature effect becomes irrelevant. For the sceptical reader we include in Fig. 3 also some points obtained numerically be using $f_\pm^{\rm TBA}(\theta,T)$ instead of $f_\pm^{\rm free}(\theta,T)$ in (4.14). The two different cases may hardly be distinguished.

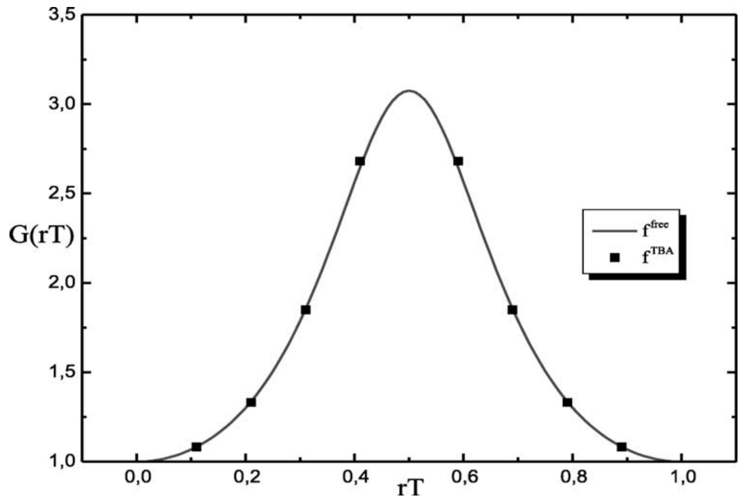

Fig. 3. Exact correlation function versus dressed one particle form factor contribution $G(rT) := \langle \overline{T}(r)\overline{T}(0)\rangle_{m=0,T\neq 0}/\langle \overline{T}(r)\overline{T}(0)\rangle^{(1)}_{m=0,T\neq 0}$.

Let us see whether the next order contributions can improve the situation. For this we have to compute

$$\begin{aligned}
&\big\langle \overline{T}(r)\overline{T}(0)\big\rangle^{(2)}_{m=0,T} \\
&\quad = \int \frac{d\theta_1\, d\theta_2}{2(2\pi)^2}\Big[f_+(\theta_1,T) f_+(\theta_2,T)\big|F^{\overline{T}}_{2,RR}(\theta_1,\theta_2)\big|^2 e^{-rT(\varepsilon(\theta_1,T)+\varepsilon(\theta_2,T))} \\
&\qquad + f_-(\theta_1,T) f_-(\theta_2,T)\big|F^{\overline{T}}_{2,RR}(\theta_1,\theta_2)\big|^2 e^{rT(\varepsilon(\theta_1,T)+\varepsilon(\theta_2,T))} \\
&\qquad + 2 f_+(\theta_1,T) f_-(\theta_2,T)\big|F^{\overline{T}}_{2,RR}(\theta_1,\theta_2-i\pi)\big|^2 e^{-rT(\varepsilon(\theta_1,T)-\varepsilon(\theta_2,T))}\Big].
\end{aligned} \tag{4.16}$$

We have already all the ingredients to compute this apart from the particle–hole form factor $F^{\overline{T}}_{2,RR}(\theta_1, \theta_2 + i\pi)$. We compute

$$F^{\overline{T}}_{2,RR}(\theta_1,\theta_2-i\pi) = -\frac{\pi}{2\nu^8}\left(\frac{2\cosh\theta_{12}+1}{2\cosh\theta_{12}-1}\right)\frac{\tanh^2(\theta_{12}/2)}{F_{\min}(\theta_{12})} e^{\theta_1+\theta_2}. \tag{4.17}$$

Assembling all we find similar values in the entire range of $0 < rT < 1$

$$\big\langle \overline{T}(r)\overline{T}(0)\big\rangle^{(2)}_{m=0,T} \Big/ \big\langle \overline{T}(r)\overline{T}(0)\big\rangle_{m=0,T} \sim 0.001 \pm 4\times 10^{-4}, \tag{4.18}$$

independently of the different choices for the dressing functions. Thus, assuming that the convergence of the series in (2.13) does not change radically when the temperature is switched on, the higher order n-particle contributions will not rescue the proposal for this case.

5. Conclusions

We provided more evidence which supports the proposal of LeClair, Lesage, Sachdev and Saleur [6] to use dressed form factors for the computation of two-point correlation functions. The method seems to work well for the free Fermion case and in addition for theories with anyonic statistics. Concerning dynamically interacting theories we reach a similar conclusion as drawn in [8] on the base of an example involving a chemical potential: namely, that it fails to work. As a simple counter example we have studied the scaling Yang–Lee model. This conclusion is reached independently of the choices for the dressing functions $f(\varepsilon_{\mathrm{TBA}}(\theta, T))$ or $f(\varepsilon_{\mathrm{free}}(\theta, T))$.

Despite this slightly pessimistic result concerning the proposal in its present form, it was shown in [6] that it also works for the computation of correlation functions in the presence of boundaries, albeit only for the Ising model. For interacting theories one can resort in the boundary set up to a modified approach [11], which replaces reflection amplitudes by renormalized ones. Based on this result one may conjecture that it can also be successfully applied to defect systems [29]. In fact in this context the only interesting integrable theories are those for which the approach seems to work for the bulk theories, namely free theories, possibly with anyonic statistical interaction. It was shown recently [30], that these theories are the only bulk theories which allow a simultaneous occurrence of reflection and transmission.

As was argued in [12], possibly nondiagonal theories still allow the application of the dressed form factor approach. However, in there only indirect evidence was provided in the low temperature regime. As may be seen from our analysis, see Fig. 3, also in the diagonal case in this regime the approach works reasonably. Thus, it remains to be established whether in generality the nondiagonal nature of the model provides a way out of the failure of the proposal.

At present the challenge remains to show whether the form factor approach to compute correlation functions can be extended successfully to the finite temperature regime in complete generality, namely also for dynamically interacting theories. In order to complete this task it would also be interesting to compare with existing alternative approaches, e.g., [31].

Acknowledgements

We are grateful to the Deutsche Forschungsgemeinschaft (Sfb288) and INTAS project 99-01459 for financial support and to M. Karowski, A. LeClair, I.T. Todorov for useful comments.

Appendix A

In this appendix we assemble some properties of various functions which are important for our computations. Some of them are standard whereas others are specific to the present context. One of the most ubiquitous functions in this context are the modified Bessel functions, whose integral representations are given by

$$K_\alpha(z) = \int_0^\infty dt\, \exp(-z\cosh t)\cosh \alpha t \quad \text{for } |\arg z| < \frac{\pi}{2}. \tag{A.1}$$

We recall the well-known limiting behaviour

$$\lim_{x\to 0} K_\alpha(x) \sim 2^{\alpha-1}\Gamma(\alpha)x^{-\alpha}, \qquad \operatorname{Re}\alpha > 0, \qquad \lim_{x\to 0} K_0(x) \sim -\ln x. \tag{A.2}$$

It is convenient to introduce the function

$$\widehat{K}_\alpha^\pm(m,r,T) = \sum_{n=0}^\infty (-1)^n \left[K_\alpha\left(\frac{nm}{T} + rm\right) \pm K_\alpha\left(\frac{(n+1)m}{T} - rm\right)\right], \tag{A.3}$$

which will appear as a building block in the computation of many finite temperature correlation functions. Since we intend to investigate the commutativity of the diagram in Fig. 1, various limits of this function will be required frequently. For $\operatorname{Re}\alpha > 0$ we compute with (A.2) the massless limit

$$\lim_{m\to 0} \widehat{K}_\alpha^\pm(m,r,T) \sim \frac{\Gamma(\alpha)}{2^{1-\alpha}} \sum_{n=0}^\infty (-1)^n \left[\left(\frac{nm}{T} + rm\right)^{-\alpha} \pm \left(\frac{(n+1)m}{T} - rm\right)^{-\alpha}\right]. \tag{A.4}$$

The sum can be evaluated explicitly. We just report on the cases which are important for our analysis

$$\lim_{m\to 0} \widehat{K}_1^+(m,r,T) \sim \frac{\pi T}{m}\frac{1}{\sin(\pi r T)}, \tag{A.5}$$

$$\lim_{m\to 0} \widehat{K}_2^-(m,r,T) \sim \left(\frac{\pi T}{m}\right)^2 \frac{2\cot(\pi r T)}{\sin^2(\pi r T)}, \tag{A.6}$$

$$\lim_{m\to 0} \widehat{K}_3^+(m,r,T) \sim \left(\frac{\pi T}{m}\right)^3 \frac{6 + 2\cos(2\pi r T)}{\sin^3(\pi r T)}, \tag{A.7}$$

$$\lim_{m\to 0} m^4 \widehat{K}_0^-(m,r,T) \sim 0, \tag{A.8}$$

$$\lim_{m\to 0} m^4 \widehat{K}_0^+(m,r,T)\widehat{K}_2^+(m,r,T) \sim 0. \tag{A.9}$$

The zero temperature limit is more easily computed, just by noting that in the sum of (A.3) only the $n = 0$ term survives

$$\lim_{T\to 0} \widehat{K}_\alpha^\pm(m,r,T) \sim K_\alpha(rm). \tag{A.10}$$

A further function which frequently occurs is Lerch's transcendental function, whose sum representation is

$$\Phi(x,s,\alpha) = \sum_{n=0}^\infty \frac{x^n}{(n+\alpha)^s}, \quad \text{for } |x| < 1,\ \alpha \notin \mathbb{Z}_0^-. \tag{A.11}$$

In many cases, however, we require precisely the value $x \to -1$ in (A.11). The convergence problem can be circumvented by exploiting the fact that in the limit $x \to -1$ we can express $\Phi(x, s, \alpha)$ in terms of Riemann zeta functions

$$\zeta(s, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}, \quad \text{for } \operatorname{Re} s > 1, \tag{A.12}$$

instead

$$\lim_{x \to -1} \Phi(x, s, \alpha) = \frac{1}{2^s}\big[\zeta(s, \alpha/2) - \zeta(s, (1+\alpha)/2)\big]. \tag{A.13}$$

This leaves the problem $\lim_{x \to -1} \Phi(x, s, \alpha)$ for $\operatorname{Re} s < 1$. Also the limit

$$\lim_{\alpha \to 0} \Phi\big(-e^{-1/\alpha}, s, \alpha\big) \sim \alpha^{-s}, \tag{A.14}$$

is needed in Section 3.

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