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Non-crystallographic reduction of generalized Calogero–Moser models

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Abstract

We apply a recently introduced reduction procedure based on the embedding of non-crystallographic Coxeter groups into crystallographic ones to Calogero–Moser systems. For rational potentials the familiar generalized Calogero Hamiltonian is recovered. For the Hamiltonians of trigonometric, hyperbolic and elliptic types, we obtain novel integrable dynamical systems with a second potential term which is rescaled by the golden ratio. We explicitly show for the simplest of these non-crystallographic models, how the corresponding classical equations of motion can be derived from a Lie algebraic Lax pair based on the larger, crystallographic Coxeter group.

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1. Introduction

The generalized Calogero–Moser models [1–12] constitute one of the most prominent and widely studied class of classical and quantum integrable systems describing ℓ particles moving on a line. In this work, we will concentrate on the classical mechanics models. Given a set of dynamical variables in terms of a coordinate vector $q \in \mathbb{R}^\ell$ and the canonically conjugate momenta $p \in \mathbb{R}^\ell$ the classical Calogero–Moser Hamiltonians take the following forms:

$$H = \frac{p^2}{2} + \frac{1}{2}\sum_{\alpha\in\Delta} g_\alpha^2 V(\alpha\cdot q), \qquad V(u) = \begin{cases} 1/u^2, & \text{rational} \\ 1/\sin^2 u, & \text{trigonometric} \\ 1/\sinh^2 u, & \text{hyperbolic} \\ 1/\mathrm{sn}^2 u, & \text{elliptic.} \end{cases} \tag{1.1}$$

The sum in the potential runs over a root system $\Delta \subset \mathbb{R}^\ell$ associated with a finite Coxeter group $\mathcal{W}$; see for instance [13] for details. g_α are coupling constants which must at least coincide on vectors α of the same length, i.e. $g_\alpha = g_\beta$ for $\alpha^2 = \beta^2$, if one demands the Hamiltonian to be invariant under $\mathcal{W}$. In the literature, the elliptic potential in (1.1) is often also expressed in

terms of the Weierstrass $\wp$-function, both choices only differ by a rescaling of the argument and an additive constant [14, 15]. In fact, all other types of potentials can be obtained from the elliptic one through special limits. In the rational case, one often also adds a confining harmonic potential $\sim q^2$ which we omit from our discussion for simplicity.

If the root system Δ is crystallographic, i.e. $2(\alpha\cdot\beta)/\beta^2 \in \mathbb{Z}$ for any pair of roots $\alpha, \beta \in \Delta$, the associated Coxeter group $\mathcal{W}$ is connected with semi-simple Lie algebras [16]. Integrability of the Calogero–Moser models (1.1) can be proved via the standard technique of Lax pairs $\{L, M\}$ [17]: if the classical equations of motion resulting from (1.1) are equivalent to the Lax equation $\dot{L} = [L, M]$, the quantities $I_k = \mathrm{Tr}\, L^k$ are conserved. In contrast to other integrable models associated with Lie algebras, as for instance (affine) Toda models [18–20], a generic Lie algebraic formulation of the Lax pair is missing and a variety of alternative approaches have been put forward in the literature; see e.g. [9, 11, 21–24]. In this paper, we will only use a Lie algebraic Lax construction for the A_ℓ or $su(\ell+1)$ series in order to exemplify our reduction procedure for the simplest model. However, we stress that in the aforementioned literature Lax pair constructions have been carried out for all algebras and for all four types of potentials in (1.1).

For the non-crystallographic Coxeter groups, $\tilde{\mathcal{W}} = I_2(m), H_3, H_4$, with root systems $\tilde{\Delta}$ one has in general $2(\tilde{\alpha}\cdot\tilde{\beta})/\tilde{\beta}^2 \notin \mathbb{Z}$ and the connection with Lie algebras ceases to be valid. Due to the latter fact the Lax construction now becomes even more difficult and only the rational potential in (1.1) has been considered using an alternative formulation based on reflection operators [24]. Other concepts such as 'exact solvability' [25, 26], based on the computation of invariants, also run into problems for non-rational potentials; see the discussion of an exactly solvable Sutherland model based on H_3 in [27].

In this paper, we overcome these difficulties by introducing for the root systems $\tilde{\Delta}$ of the non-crystallographic Coxeter groups $H_2 \equiv I_2(5), H_3, H_4$ an extension of the Calogero–Moser Hamiltonian (1.1) which allows us to tie the proof of integrability for *all* four types of potentials to that of certain crystallographic groups specified below (1.4). Namely, we consider the Hamiltonians

$$\tilde{H} = \frac{\tilde{p}^2}{2} + \frac{\tilde{g}^2}{2}\sum_{\tilde{\alpha}\in\tilde{\Delta}}\{V(\tilde{\alpha}\cdot\tilde{q}) + V(\phi\tilde{\alpha}\cdot\tilde{q})\}, \qquad \phi = \phi^2 - 1 = \frac{1+\sqrt{5}}{2}. \tag{1.2}$$

The constant ϕ entering the second potential term is the well-known golden ratio[1]. Clearly, in the case of the rational potential adding the extra term in (1.2) amounts to a simple rescaling of the coupling constant in (1.1) and we have

$$V(u) = u^{-2} \qquad \tilde{H}(\tilde{g}^2) = H((1+\phi^{-2})\tilde{g}^2). \tag{1.3}$$

Thus, we recover the familiar generalization of the Calogero model to non-crystallographic root systems $\tilde{\Delta}$. For the remaining cases, the insertion of the extra potential term might appear ad hoc at first sight but we will explain in the text that it occurs naturally in light of the following embeddings of non-crystallographic Coxeter groups into crystallographic ones:

$$H_2 \hookrightarrow A_4, \qquad H_3 \hookrightarrow D_6 \qquad \text{and} \qquad H_4 \hookrightarrow E_8. \tag{1.4}$$

Employing a reduction procedure recently introduced in the context of affine Toda field theory [28] the models (1.2) are obtained from the Hamiltonians (1.1) of the corresponding crystallographic groups given in (1.4). See also [29] and references therein for similar reduction procedures. Close analogies also exist with the folding procedure of crystallographic

[1] Instead of ϕ one can also use its inverse ϕ^{-1}. Since this change does not affect the defining relations of the respective non-crystallographic Coxeter groups [13] we may restrict our discussion to the common choice ϕ without loss of generality.

Coxeter groups linked with simply laced algebras into those corresponding to non-simply laced ones (see [20, 30] in the context of affine Toda and [31] for Calogero–Moser models). However, there are several differences in the mathematical structure. We will comment on this further in the text.

Our main result in this paper is the extension of the non-crystallographic Calogero–Moser models from the rational case treated so far to trigonometric, hyperbolic and elliptic potentials. However, our reduction procedure also puts a new perspective on the familiar rational case, it enables one to connect the non-crystallographic models to Lie algebras through the embeddings (1.4). The proof of Liouville integrability of the models (1.2) can be carried out by employing the structure of the Lax pairs associated with the crystallographic root systems in (1.4). We explicitly demonstrate this for the simplest model, the one associated with H_2, by exploiting a known Lie algebraic Lax pair related to $A_4 \equiv su(5)$, despite the fact that one is dealing with a non-crystallographic Coxeter group on the level of the Hamiltonian.

The paper is structured as follows. In section 2, we review the embeddings (1.4) and introduce the necessary mathematical formalism for our reduction procedure. In section 3, it is then explained how to reduce the crystallographic Hamiltonians (1.1) and the associated equations of motion to the non-crystallographic systems (1.2). We address the question of integrability in section 4 by showing the existence of a Lax pair for the simplest model associated with H_2. This Lax pair is obtained through the reduction of a Lie algebraic pair for A_4. Comparison with other Lax pair formulations [23, 24] is made in the appendix. Section 5 contains our conclusions.

2. Embedding of non-crystallographic into crystallographic Coxeter groups

The details of the embeddings (1.4) have been presented previously in the literature [32–34] and in particular in [28] which we follow in our notation. We therefore omit proofs and only present the necessary formulae for the reduction. Throughout this paper quantities related to the two different types of Coxeter groups in (1.4) will be distinguished by putting an additional tilde on top of the non-crystallographic quantities. In light of (1.4), we henceforth limit ourselves to the simply laced case.

Recall [13] that any Coxeter group $\mathcal{W}$ is generated by the reflections associated with a set of simple roots $\{\alpha_i\} \subset \Delta$:

$$\sigma_i(x) = x - 2\frac{x \cdot \alpha_i}{\alpha_i \cdot \alpha_i}\alpha_i \qquad \text{for} \quad 1 \leqslant i \leqslant \ell, \quad x \in \mathbb{R}^\ell. \tag{2.1}$$

This set of reflections generates the Coxeter group $\mathcal{W}$ subject to the relations

$$(\sigma_i\sigma_j)^{m_{ij}} = 1, \qquad 1 \leqslant i, \quad j \leqslant \ell, \tag{2.2}$$

where the order $m_{ij} \in \mathbb{N}$ of the group elements is defined through the Cartan matrix K,

$$m_{ij} = \pi/\arccos(-K_{ij}/2), \qquad K_{ij} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_j \cdot \alpha_j}. \tag{2.3}$$

Relations (2.1), (2.2) and (2.3) apply also to the non-crystallographic group $\tilde{\mathcal{W}}$. Since we are only dealing with root systems $\Delta, \tilde{\Delta}$ where all elements have equal length, we adopt henceforth the normalization convention $\alpha_i^2 = \tilde{\alpha}_i^2 = 2$.

Introducing a special labelling of the simple roots $\{\alpha_i\}$ depicted in figure 1 allows for combining the different embeddings in (1.4) into a single formula [28]:

$$\tilde{\mathcal{W}} \hookrightarrow \mathcal{W}: \qquad \tilde{\sigma}_i \mapsto \sigma_i\sigma_{i+\tilde{\ell}} = \sigma_{i+\tilde{\ell}}\sigma_i \qquad \text{for} \quad 1 \leqslant i \leqslant \tilde{\ell}. \tag{2.4}$$

Note that the rank $\tilde{\ell}$ of the non-crystallographic group and the rank ℓ of its crystallographic counterpart in (1.4) are always related by a factor 2, $\ell = 2\tilde{\ell}$. Furthermore, our labelling of

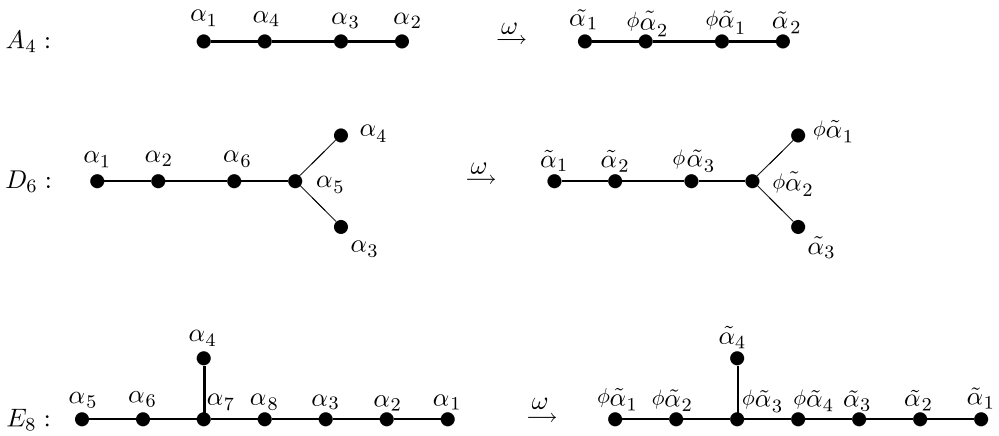

Figure 1. Coxeter graphs, root labelling and the map (2.7) for the Coxeter groups in (1.4).

the simple roots is such that the roots $\alpha_i, \alpha_{i+\tilde{\ell}}$ are always orthogonal whence the associated reflections commute. The embedding (2.4) is to be understood in the sense of a group homomorphism, i.e. it preserves the Coxeter relations (2.2).

In order to realize (2.4) in the context of the Calogero–Moser models (1.1), we need to know how this embedding manifests itself on the level of the corresponding root spaces Δ and $\tilde{\Delta}$ which are left invariant by $\mathcal{W}$ and $\tilde{\mathcal{W}}$, respectively. This is achieved by defining a pair of maps

$$\omega : \Delta \to \tilde{\Delta} \cup \phi\tilde{\Delta} \qquad \text{and} \qquad \tilde{\omega} : \tilde{\Delta} \to \Delta \oplus \phi\Delta, \tag{2.5}$$

which intertwine the embedding (2.4), i.e.

$$\tilde{\sigma}_i \omega = \omega \sigma_i \sigma_{i+\tilde{\ell}} \qquad \text{and} \qquad \tilde{\omega}\tilde{\sigma}_i = \sigma_i \sigma_{i+\tilde{\ell}} \tilde{\omega}. \tag{2.6}$$

The first map ω has been previously considered in the literature, see e.g. [28, 33], and is defined as follows:

$$\alpha_i \mapsto \omega(\alpha_i) = \begin{cases} \tilde{\alpha}_i & \text{for} \quad 1 \leqslant i \leqslant \tilde{\ell} = \ell/2 \\ \phi\tilde{\alpha}_{i-\tilde{\ell}} & \text{for} \quad \tilde{\ell} < i \leqslant \ell. \end{cases} \tag{2.7}$$

The second map $\tilde{\omega}$, introduced in [28] and paramount to our reduction procedure, realizes the simple root system $\{\tilde{\alpha}_i\}$ of the non-crystallographic Coxeter group $\tilde{\mathcal{W}}$ in $\mathbb{R}^\ell$ by identifying

$$\tilde{\alpha}_i \mapsto \tilde{\omega}(\tilde{\alpha}_i) = \alpha_i + \phi\alpha_{i+\tilde{\ell}} \qquad \text{for} \quad 1 \leqslant i \leqslant \tilde{\ell}. \tag{2.8}$$

The images of the non-crystallographic roots have now length $\tilde{\omega}(\tilde{\alpha}_i)^2 = 2(1+\phi^2)$ according to our earlier convention $\tilde{\alpha}_i^2 = 2$. Thus, $\tilde{\omega}$ preserves the inner product only up to a factor $(1+\phi^2)$.

As the simple roots $\{\alpha_i\}$ and $\{\tilde{\alpha}_i\}$ are linearly independent the maps (2.7) and (2.8) can be linearly extended to the whole vector spaces $\mathbb{R}^\ell$ respectively $\mathbb{R}^{\tilde{\ell}}$. We will make use of this fact when reducing the crystallographic Calogero–Moser systems to non-crystallographic ones below. Note that the defining relations (2.7) and (2.8) also apply to the fundamental weights [28].

Using the pair $\omega, \tilde{\omega}$ we are in the position to relate inner products in $\tilde{\Delta}$ to inner products in Δ by means of the identity

$$\omega(\alpha_i) \cdot \tilde{\alpha}_j = \alpha_i \cdot \tilde{\omega}(\tilde{\alpha}_j) \qquad \text{for} \quad 1 \leqslant j \leqslant \tilde{\ell}, \quad 1 \leqslant i \leqslant \ell. \tag{2.9}$$

From this relationship as well as (2.5) and (2.6), we infer that $\tilde{\omega}$ plays the role of a 'quasi-inverse' to the map ω, in fact we have that

$$\omega\tilde{\omega} = (1+\phi^2)\mathbb{I} \qquad \text{and} \qquad \tilde{\omega}\omega = \begin{pmatrix} \mathbb{I} & \phi\mathbb{I} \\ \phi\mathbb{I} & \phi^2\mathbb{I} \end{pmatrix}, \tag{2.10}$$

with $\mathbb{I}$ denoting the $\tilde{\ell} \times \tilde{\ell}$ identity matrix. As an immediate consequence of (2.9) we obtain a crucial relationship between the non-crystallographic Cartan matrix $\tilde{K}$ and the crystallographic one K. Namely, introducing the $\tilde{\ell} \times \tilde{\ell}$ matrices κ and $\hat{\kappa}$ through the following block decomposition of the crystallographic Cartan matrix

$$K = \begin{pmatrix} \kappa & \hat{\kappa} \\ \hat{\kappa} & \kappa + \hat{\kappa} \end{pmatrix}, \tag{2.11}$$

we have the matrix equation

$$\tilde{K} = \kappa + \phi\hat{\kappa} = \phi^{-1}\hat{\kappa} + \kappa + \hat{\kappa}. \tag{2.12}$$

Employing definitions (2.7) and (2.8) together with identities (2.9) and (2.12) the intertwining relations (2.6) now follow from a straightforward computation. Similar identities also hold for the inverse Cartan matrices [28].

3. Reduction of crystallographic Calogero–Moser models

Having introduced the necessary mathematical set-up we are now in the position to introduce our reduction map. We start from a dynamical system defined in terms of the Hamiltonian (1.1) based on any of the three crystallographic Coxeter groups in (1.4). Such a system depends on ℓ independent dynamical variables $q = (q_1, \ldots, q_\ell)$ and ℓ independent conjugate momenta $p = (p_1, \ldots, p_\ell)$. We now replace this set of variables by a new one which only contains $\tilde{\ell}$ independent coordinates and $\tilde{\ell}$ independent momenta by defining the following reduction map μ:

$$(q, p) \to (\mu(q), \mu(p)) := (\tilde{\omega}(\tilde{q}), \tilde{\omega}(\tilde{p})). \tag{3.1}$$

Here the action of $\tilde{\omega}$ on the simple roots $\tilde{\alpha}_i$ is defined in (2.8). The vectors $\tilde{q} = (\tilde{q}_1, \ldots, \tilde{q}_{\tilde{\ell}})$ and $\tilde{p} = (\tilde{p}_1, \ldots, \tilde{p}_{\tilde{\ell}})$ in the Euclidean basis will become the dynamical variables with respect to the non-crystallographic Hamiltonian (1.2) resulting from the reduced Hamiltonian:

$$H(q, p) \overset{\mu}{\to} H^{\text{red}} := H(\tilde{\omega}(\tilde{q}), \tilde{\omega}(\tilde{p})). \tag{3.2}$$

Our particular choice for the definition of the reduction map (3.1) in terms of the map $\tilde{\omega}$ will allow us to discuss the reduction procedure without making reference to a specific representation of the root spaces Δ and $\tilde{\Delta}$. By exploting the identity (2.9), the reduction (3.1) can be carried out in terms of the root systems instead of the dynamical variables and momenta.

Let us further motivate (3.1) by comparing it to the reduction when folding a simply laced Lie algebra by a non-trivial Dynkin diagram automorphism τ to a non-simply laced algebra [20]. In that context the reduction occurs when the coordinates q and momenta p are projected onto the invariant subspaces under τ. This decreases the number of independent variables. The reduced or folded Calogero–Moser Hamiltonian [31] is then obtained by inserting the projected variables into the original 'simply laced' Hamiltonian (1.1) and rewriting it in terms of the τ-invariant root subspace $\Delta_\tau \subset \Delta$. The latter can be identified with the root system Δ^{ns} of a non-simply laced algebra. We will comment further on this analogy below; see (3.10).

We proceed here analogously and now explain how the reduced Hamiltonian (3.2) can be expressed in terms of the non-crystallographic root system $\tilde{\Delta}$ only.

3.1. The reduced Hamiltonian

As may be seen in (1.4) all the Lie algebras relevant to our reduction procedure are simply laced and we set $g_\alpha = g$ in (1.1). Then the reduced Hamiltonian (3.2) can be rewritten as follows:

$$2H^{\text{red}} = \tilde{\omega}(\tilde{p})^2 + g^2 \sum_{\alpha\in\Delta} V(\alpha\cdot\tilde{\omega}(\tilde{q})) \tag{3.3}$$

$$= (1+\phi^2)\tilde{p}^2 + g^2 \sum_{\alpha\in\Delta} V(\omega(\alpha)\cdot\tilde{q})$$

$$= (1+\phi^2)\left\{\tilde{p}^2 + \tilde{g}^2 \sum_{\tilde{\alpha}\in\tilde{\Delta}} V(\tilde{\alpha}\cdot\tilde{q}) + \tilde{g}^2 \sum_{\tilde{\alpha}\in\tilde{\Delta}} V(\phi\tilde{\alpha}\cdot\tilde{q})\right\} = 2(1+\phi^2)\tilde{H}. \tag{3.4}$$

In the last line all data belong to the non-crystallographic root system and we have arrived at (1.2). Let us first explain the reduction of the potential term. To obtain the second line we have used the inner product identity (2.9) which replaces crystallographic roots by non-crystallographic ones. Exploiting that the map ω defined in (2.7) is surjective we arrive at the third line (3.4). Here the sum over the crystallographic root system Δ is now replaced by sums over the two copies of the non-crystallographic root space $\tilde{\Delta}$ appearing in the target space of ω; compare with (2.5). In the last line we have also defined a rescaled coupling constant by setting

$$g = \sqrt{1+\phi^2}\,\tilde{g}. \tag{3.5}$$

This scaling factor arises from the kinetic energy term and is due to the fact that $\tilde{\omega}$ is not an isometry. Expanding $\tilde{p} = \sum_i \tilde{r}^i\tilde{\alpha}_i$ we compute

$$\tilde{\omega}(\tilde{p})\cdot\tilde{\omega}(\tilde{p}) = (1+\phi^2)\sum_{i,j=1}^{\tilde{\ell}} \tilde{r}^i(\kappa_{ij}+\phi\hat{\kappa}_{ij})\tilde{r}^j = (1+\phi^2)\sum_{i,j=1}^{\tilde{\ell}} \tilde{r}^i\tilde{K}_{ij}\tilde{r}^j = (1+\phi^2)\tilde{p}^2, \tag{3.6}$$

where $\kappa, \hat{\kappa}$ have been defined in (2.11). Furthermore, we have used the normalization convention $\tilde{\alpha}_i^2 = \alpha_i^2 = 2$.

3.2. Invariance under the non-crystallographic Coxeter group

It is apparent from the explicit form of the Hamiltonian (3.4) that the reduced Calogero–Moser model is invariant under the non-crystallographic Coxeter group $\tilde{\mathcal{W}}$. Employing the intertwining property (2.6) and the fact that Coxeter transformations preserve the inner product, we see that the action of $\tilde{\mathcal{W}}$ in the 'crystallographic variant' (3.3) of the new Hamiltonian (1.2) is realized through the embedding (2.4). For instance, we have for the potential

$$\sum_{\alpha\in\Delta} V(\tilde{\omega}(\tilde{\sigma}_i\tilde{q})\cdot\alpha) = \sum_{\alpha\in\Delta} V(\sigma_i\sigma_{i+\tilde{\ell}}\tilde{\omega}(\tilde{q})\cdot\alpha) = \sum_{\alpha\in\Delta} V(\tilde{\omega}(\tilde{q})\cdot\alpha). \tag{3.7}$$

A similar identity holds for the kinetic term. We can use this fact to show that the coupling constants in front of the two potential terms in (3.4) can be chosen independently without violating invariance under the non-crystallographic Coxeter group $\tilde{\mathcal{W}}$. This is apparent from the variant (3.4), but as a preparatory step for the reduction of a crystallographic Lax pair below it is instructive to directly verify this also in terms of the reduced Hamiltonian (3.3).

First, we need to split the crystallographic root system Δ into the following disjoint subsets:

$$\Delta = \Delta' \cup \Delta'' \qquad \text{with} \qquad \omega(\Delta') = \tilde{\Delta} \qquad \text{and} \qquad \omega(\Delta'') = \phi\tilde{\Delta}. \tag{3.8}$$

In order to see that these sets are indeed disjoint, note that for any root $\tilde{\alpha} \in \tilde{\Delta}$ the vector $\phi\tilde{\alpha} \notin \tilde{\Delta}$. Otherwise, there had to be a group element $\tilde{w} \in \tilde{\mathcal{W}}$ which maps $\tilde{\alpha}$ into $\phi\tilde{\alpha}$, as the action of the Coxeter group exhausts the entire root space. If such an element would exist, we had the identity $\tilde{w}(\tilde{\alpha})^2 = \phi^2\tilde{\alpha}^2 \neq 2$ which contradicts the fact that the Coxeter group $\tilde{\mathcal{W}}$ preserves the inner product. We can therefore conclude that $\tilde{\Delta} \cap \phi\tilde{\Delta} = \varnothing$ and therefore $\Delta' \cap \Delta'' = \varnothing$. This then implies that Δ' and Δ'' must be left invariant under the action of the non-crystallographic Coxeter group with respect to the embedding (2.4). Namely, according to (2.6) we have for any root $\alpha' \in \Delta'$ that

$$\tilde{\sigma}_i\omega(\alpha') = \omega(\sigma_i\sigma_{i+\tilde{\ell}}\alpha') \in \tilde{\Delta},$$

which entails that $\sigma_i\sigma_{i+\tilde{\ell}}\Delta' = \Delta'$ for all $i = 1, \ldots, \tilde{\ell}$. A similar argument holds for Δ''. Taking invariance under the Coxeter group $\tilde{\mathcal{W}}$ as a guiding principle, we can therefore generalize (3.3) by introducing the following modified reduced crystallographic Hamiltonian:

$$H^{\mathrm{red}} = \frac{\tilde{\omega}(\tilde{p})^2}{2} + \frac{g_1^2}{2}\sum_{\alpha'\in\Delta'} V(\alpha'\cdot\tilde{\omega}(\tilde{q})) + \frac{g_2^2}{2}\sum_{\alpha''\in\Delta''} V(\alpha''\cdot\tilde{\omega}(\tilde{q})), \tag{3.9}$$

where g_1, g_2 are now arbitrary. The appearance of an additional free coupling constant in the reduction is very reminiscent of the folding procedure [20] in the context of Calogero–Moser models [31] already mentioned previously.

3.3. Comparison with folding

The structure of the Calogero–Moser Hamiltonian (3.9) is similar to that obtained by folding a simply laced root system Δ into a non-simply laced one Δ^{ns} via a Dynkin diagram automorphism τ. The potential term in the 'folded' Hamiltonian [31] also splits into two parts:

$$H^{\mathrm{ns}} = \frac{p^2}{2} + \frac{g_s^2}{2}\sum_{\alpha\in\Delta_s} V(\alpha\cdot q) + \frac{g_l^2}{2}\sum_{\alpha\in\Delta_l} V(\alpha\cdot q), \tag{3.10}$$

one running over the short roots Δ_s, the other over the long roots Δ_l, each constituting an independent Weyl group orbit. Note the absence of the scaling factor in the second term in comparison to (3.9).[2] Similar as in our initial calculation leading to (3.4) the coupling constants g_s, g_l are not independent but related by $|\tau|$ due to the folding procedure. However, outside the framework of folding one can often choose the two couplings independently without violating integrability [9, 11, 23, 29, 31]. An exception in the framework of Lie algebraic Lax pairs is the B_ℓ series [29] and G_2 [36]. We will verify below whether we can retain in the present context the two independent couplings in (3.9) for the reduction of a Lie algebraic Lax pair.

3.4. The equations of motion

Before the construction of a Lax pair we first apply our reduction procedure to the classical equations of motion. Let ∇_q denote the gradient operator with respect to the Euclidean basis in q-space ($\cong \mathbb{R}^\ell$). Then the equations of motion originating from the crystallographic variant (3.9) of the Hamiltonian are

$$\tilde{\omega}(\tilde{p}) = \tilde{\omega}(\dot{\tilde{q}}) \qquad \text{and} \qquad \tilde{\omega}(\dot{\tilde{p}}) = -\nabla_q H^{\mathrm{red}} = -\frac{1}{2}\sum_{\alpha\in\Delta} \alpha g_\alpha^2 V'(\alpha\cdot\tilde{\omega}(\tilde{q})), \tag{3.11}$$

[2] We only mention here the case which has been referred to as 'untwisted' in [31], i.e. τ is an automorphism related to the non-extended Dynkin diagram; see [31, 35] for other possibilities.

where we set $g_\alpha = g_1$ for $\alpha \in \Delta'$ and $g_\alpha = g_2$ for $\alpha \in \Delta''$. Acting on both sides of these two equations with the map ω defined in (2.7) together with the identities (2.9) and (2.10) we obtain the reduced system

$$\tilde{p} = \dot{\tilde{q}} \qquad \text{and} \qquad \dot{\tilde{p}} = -\nabla_{\tilde{q}} \tilde{H} = -\frac{1}{2} \sum_{\tilde{\alpha} \in \tilde{\Delta}} \left\{ \tilde{\alpha} \tilde{g}_1^2 V'(\tilde{\alpha} \cdot \tilde{q}) + \phi \tilde{\alpha} \tilde{g}_2^2 V'(\phi \tilde{\alpha} \cdot \tilde{q}) \right\} \tag{3.12}$$

corresponding to the non-crystallographic Hamiltonian (3.4). Here $\tilde{g}_i = g_i/(1+\phi^2)^{1/2}$, $i = 1, 2$ and $\nabla_{\tilde{q}}$ is now the gradient operator with respect to the Euclidean basis in $\tilde{q}$-space ($\cong \mathbb{R}^{\tilde{\ell}}$). Note that system (3.11) is more restrictive than (3.12), i.e. any solution to (3.11) yields a solution of (3.12) but the converse is not necessarily true. To see this, one can apply the map $\tilde{\omega}$ on both sides of (3.12) using that $\tilde{\omega}(\tilde{\alpha}) = \alpha' + \phi\alpha''$ with $\tilde{\alpha} = \omega(\alpha') = \phi^{-1}\omega(\alpha'')$. A similar observation applies in the context of folding.

The first crucial step to show integrability of our reduced systems is to show that (3.12) can be equivalently formulated in terms of a Lax pair. In this context, our ability to express the non-crystallographic Hamiltonian (3.4) and the equations of motion (3.12) in crystallographic terms, (3.3) and (3.11), will be essential.

4. A Lie algebraic Lax pair for the H_2 model

As pointed out in the introduction there is no generic Lie algebraic formulation for the Lax pairs of the generalized Calogero–Moser models. Instead a variety of different constructions for Lax pairs have been put forward in the literature, see e.g. [9, 11, 21–24, 29], which will not be discussed in detail. We focus on the simplest model associated with H_2, where according to (1.4) the corresponding crystallographic system is related to A_4 respectively $su(5)$. The original construction of the Lax pair for the A_ℓ series goes back to Calogero [21]. We shall adopt here its formulation in the Cartan–Weyl basis (as it can be found for instance in [9, 29, 37]), since in this setting the computation is more general. For the moment we keep the rank ℓ arbitrary, such that one can easily adopt our discussion to the cases in (1.4) omitted here. We shall specialize to the relevant case $\ell = 4$ below.

Consider the Cartan–Weyl basis defined through the commutation relations (e.g. [16])

$$[H_i, E_\alpha] = \alpha^i E_\alpha, \qquad [E_\alpha, E_{-\alpha}] = \alpha \cdot H, \qquad [E_\alpha, E_\beta] = \varepsilon_{\alpha,\beta} E_{\alpha+\beta}. \tag{4.1}$$

In the last commutator it is understood that $\alpha \neq -\beta$ and $\varepsilon_{\alpha,\beta} = 0$ whenever $\alpha + \beta$ is not an element of the root space Δ. The compatible choice of the trace is

$$\mathrm{Tr}(H_i H_j) = \delta_{ij} \qquad \text{and} \qquad \mathrm{Tr}(E_\alpha E_{-\alpha}) = 1, \tag{4.2}$$

which implies together with our previous convention $\alpha^2 = 2$ that the structure constants $\varepsilon_{\alpha,\beta}$ only assume the values $0, \pm 1$. In addition, they satisfy the following general identities:

$$\varepsilon_{\alpha,\beta} = -\varepsilon_{\beta,\alpha} = -\varepsilon_{-\alpha-\beta,\beta}. \tag{4.3}$$

We require $E_\alpha^\dagger = E_{-\alpha}$ which implies the further constraint $\varepsilon_{\alpha,\beta} = -\varepsilon_{-\alpha,-\beta}$. The Lax pair is now expressed in terms of the Cartan–Weyl basis as follows:

$$L = p \cdot H + \mathrm{i} \sum_{\alpha \in \Delta} g_\alpha x(\alpha \cdot q) E_\alpha \qquad \text{and} \qquad M = z \cdot H + \mathrm{i} \sum_{\alpha \in \Delta} g_\alpha x'(\alpha \cdot q) E_\alpha. \tag{4.4}$$

Here we have once more introduced the root-dependent coupling constants $g_\alpha = g_1$ for $\alpha \in \Delta'$ and $g_\alpha = g_2$ for $\alpha \in \Delta''$ in light of our reduced Hamiltonian (3.9). For the original A_ℓ Calogero–Moser model one has $g_\alpha = g$. The vector $z = z(q) \in \mathbb{R}^\ell$ in (4.4) will be specified

momentarily. Let us first define the coefficient function $x = x(u)$, which can take one of the following forms for the various types of potentials in (1.1) [9, 11, 15, 29]:

$$x(u) = \begin{cases} 1/u, & \text{rational} \\ 1/\sin u, & \text{trigonometric} \\ 1/\sinh u, & \text{hyperbolic} \\ 1/\mathrm{sn}\, u, & \text{elliptic.} \end{cases} \tag{4.5}$$

There are other possible choices for the coefficient functions [29] which can also depend on a spectral parameter [22, 24, 38]. For our disucssion of the reduction procedure we picked the present ones, because they are the simplest, but our results do also apply to the more general cases. The coefficient functions (4.5) satisfy a number of identities

$$x(u) = -x(-u), \qquad x(u)x(-u) = -V(u), \tag{4.6}$$

and crucially [9, 15]

$$x(u)x'(w) - x'(u)x(w) = [V(u) - V(w)]x(u+w). \tag{4.7}$$

Using the first two relations (4.6) one shows that

$$\mathrm{Tr}\, L^2 = p^2 - \sum_{\alpha\in\Delta} x_\alpha x_{-\alpha} = 2H \tag{4.8}$$

with x_α being shorthand notation for $x_\alpha(q) = g_\alpha x(\alpha\cdot q)$. The third identity (4.7) comes into play when showing the equivalence of the classical equations of motion to the aforementioned Lax equation:

$$\dot{L} = [L, M]. \tag{4.9}$$

Comparing the coefficients of the Cartan–Weyl basis in (4.9) one deduces

$$\dot{p}\cdot H = -\sum_{\alpha\in\Delta} \alpha\cdot H x_\alpha x'_{-\alpha} \qquad \text{and} \qquad \sum_{\alpha\in\Delta}(\alpha\cdot\dot{q})x'_\alpha E_\alpha = \sum_{\alpha\in\Delta}(p\cdot\alpha)x'_\alpha E_\alpha, \tag{4.10}$$

which are equivalent to the equations of motion $\dot{p} = -\nabla_q H$ and $\dot{q} = p$. In addition certain 'unwanted' terms must cancel which lead to a functional equation [9, 15] for the as yet unspecified vector z. This equation can be simplified using (4.7)

$$\begin{aligned} \alpha\cdot z &= \mathrm{i}\sum_{\substack{\beta,\gamma\in\Delta\\ \alpha=\beta+\gamma}} \varepsilon_{\beta,\gamma}\frac{x_\beta x'_\gamma}{x_\alpha} = \mathrm{i}\sum_{\beta\in\Delta}\varepsilon_{-\alpha,\beta}\frac{x_\beta x'_{\alpha-\beta} - x'_\beta x_{\alpha-\beta}}{x_\alpha} \\ &= -2\mathrm{i}\sum_{\beta\in\Delta}\varepsilon_{\alpha,\beta}\frac{g_\beta g_{\alpha+\beta}}{g_\alpha}V(\beta\cdot q). \end{aligned} \tag{4.11}$$

Here we have used $g_\beta = g_{-\beta}$, $V(u) = V(-u)$ as well as the symmetries $\varepsilon_{\beta,\gamma} = -\varepsilon_{\gamma,\beta} = \varepsilon_{-\alpha,\beta}$ and $\varepsilon_{-\alpha,\beta} = -\varepsilon_{-\alpha,\alpha-\beta} = -\varepsilon_{\alpha,-\beta}$ for the structure constants. It is not clear *a priori* that such a vector z always exists, but if it does, it must be unique as the functional equation (4.11) applies among others also to the simple roots $\{\alpha_i\}$ which are linearly independent. Ambiguities arise when the root space is realized as a hyperplane in a higher-dimensional space. Then we might add an arbitrary vector z' which is orthogonal on Δ, i.e. $z'\cdot\alpha = 0$ for all roots α. Employing the fundamental weights λ_i, which form the dual basis to the simple roots, $\lambda_i\cdot\alpha_j = \delta_{ij}$, it is immediate to derive that

$$z(q) = -2\mathrm{i}\sum_{\beta\in\Delta} V(\beta\cdot q)\sum_{i=1}^{\ell}\varepsilon_{\alpha_i,\beta}\frac{g_\beta g_{\alpha_i+\beta}}{g_{\alpha_i}}\lambda_i. \tag{4.12}$$

Table 1. Root decompositions and Lie algebraic structure constants for A_4.

	$(\beta_1, \varepsilon_{\alpha_i,\beta_1})$	$(\beta_2, \varepsilon_{\alpha_i,\beta_2})$	$(\beta_3, \varepsilon_{\alpha_i,\beta_3})$
α_1:	$(\alpha_2+\alpha_3+\alpha_4, +1)$	$(\alpha_3+\alpha_4, +1)$	$(\alpha_4, +1)$
α_2:	$(\alpha_1+\alpha_3+\alpha_4, -1)$	$(\alpha_3+\alpha_4, -1)$	$(\alpha_3, -1)$
α_3:	$(\alpha_1+\alpha_4, -1)$	$(\alpha_4, -1)$	$(\alpha_2, +1)$
α_4:	$(\alpha_2+\alpha_3, +1)$	$(\alpha_3, +1)$	$(\alpha_1, -1)$

For an explicit computation of the vector z and checking its consistency with (4.11) for all roots $\alpha \in \Delta$ one needs to fix the signs of the structure constants $\varepsilon_{\alpha,\beta}$ in a consistent manner. To this end we now specialize to a specific representation and set $\ell = 4$.

4.1. The Lax pair in the vector representation of A_ℓ

Let $\{e_i\}_{i=1}^5$ be the orthonormal basis in the Euclidean space $\mathbb{R}^5$. Then a standard representation of the simple roots is [13]

$$\alpha_1 = e_1 - e_2, \qquad \alpha_4 = e_2 - e_3, \qquad \alpha_3 = e_3 - e_4, \qquad \alpha_2 = e_4 - e_5. \tag{4.13}$$

Note that our labelling of the simple roots differs from the common one due to our convention for the embedding (2.4). The entire root system consists of the vectors

$$\Delta = \Delta_+ \cup -\Delta_+, \qquad \Delta_+ = \{\alpha = e_i - e_j : 1 \leqslant i < j \leqslant 5\} \tag{4.14}$$

and the Cartan–Weyl basis is given in terms of the unit matrices $(\mathfrak{e}_{ij})_{lk} = \delta_{il}\delta_{jk}$ by identifying

$$H_i = \mathfrak{e}_{ii} \qquad \text{and} \qquad E_\alpha = \mathfrak{e}_{ij} \qquad \text{if} \qquad \alpha = e_i - e_j. \tag{4.15}$$

For each simple root α_i one finds six non-vanishing structure constants $\varepsilon_{\alpha_i,\beta}$ only three of which are independent due to the symmetries $\varepsilon_{\alpha_i,\beta} = -\varepsilon_{\beta,\alpha_i} = \varepsilon_{-\alpha_i-\beta,\alpha_i} = -\varepsilon_{\alpha_i,-\alpha_i-\beta}$. Choosing β to be positive the root decompositions $\alpha = \beta + \gamma$ and the corresponding structure constants in (4.11) can be inferred from table 1.

In order to accommodate the possibility of two independent coupling constants in the reduced Hamiltonian (3.9) we need to identify the subsets (3.8). Setting $\Delta'_+ = \Delta' \cap \Delta_+$ and $\Delta''_+ = \Delta'' \cap \Delta_+$, we have

$$\Delta'_+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_4, \alpha_2+\alpha_3, \alpha_3+\alpha_4\}$$

and

$$\Delta''_+ = \{\alpha_3, \alpha_4, \alpha_1+\alpha_3+\alpha_4, \alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\alpha_4\}.$$

Making the same replacement in the dynamical variables (3.1) as in the previous reduction of the Hamiltonian (3.3) respectively (3.9), we consider the crystallographic Lax pair with coordinates

$$q \to \mu(q) = \tilde{\omega}(\tilde{q}) = (\tilde{s}_1, -\tilde{s}_1 + \phi\tilde{s}_2, \phi(\tilde{s}_1 - \tilde{s}_2), -\phi\tilde{s}_1 + \tilde{s}_2, -\tilde{s}_2). \tag{4.16}$$

Here $q = \sum_i q_i e_i$, i.e. the dynamical variables q_i are the coordinates with respect to the Euclidean basis $\{e_i\}$. $\tilde{s}_i$, on the other hand, are the components with respect to the simple roots, $\tilde{q} = \sum_i \tilde{s}_i\tilde{\alpha}_i$, and we have inserted the explicit representation (4.13) for the simple roots α_i. Note, however, that the dynamical variables $\tilde{q}_i$ are those with respect to the Euclidean basis. Both are related by a simple linear transformation, i.e. $\tilde{s}_i$ is a linear function of the Euclidean coordinates $\tilde{s}_i = \tilde{s}_i(\tilde{q}_1, \tilde{q}_2)$, whose explict form depends on the particular representation one chooses for the non-crystallographic roots. For instance, if we set [13]

$$\tilde{\alpha}_1 = \tfrac{1}{\sqrt{2}}(\phi, \sqrt{3-\phi}) \qquad \text{and} \qquad \tilde{\alpha}_2 = -\tfrac{1}{\sqrt{2}}(\phi^{-1}, \sqrt{3+\phi^{-1}}) \tag{4.17}$$

then

$$\tilde{s}_1 = \frac{\phi}{\sqrt{2}}\left(\tilde{q}_1 - \cot\frac{2\pi}{5}\tilde{q}_2\right) \qquad \text{and} \qquad \tilde{s}_2 = \frac{1}{\sqrt{2}}\left(\tilde{q}_1 - \cot\frac{\pi}{5}\tilde{q}_2\right). \tag{4.18}$$

A similar replacement holds for the conjugate momenta, $p \to \mu(p) = \tilde{\omega}(\tilde{p})$. None of the algebraic properties of the Lax pair is changed and one therefore verifies immediately that analogues of (4.8) and (4.10) hold for the reduced, non-crystallographic Hamiltonian. The latter imply the reduced crystallographic equations of motion (3.11)

$$\alpha \cdot \tilde{\omega}(\dot{\tilde{q}})x'(\omega(\alpha)\cdot\tilde{q}) = \alpha\cdot\tilde{\omega}(\tilde{p})x'(\omega(\alpha)\cdot\tilde{q}) \quad \Rightarrow \quad \tilde{\omega}(\dot{\tilde{q}}) = \tilde{\omega}(\tilde{p}) \tag{4.19}$$

and

$$\tilde{\omega}(\dot{\tilde{p}}) = \sum_{\alpha\in\Delta}\alpha x_{\omega(\alpha)}x'_{-\omega(\alpha)} = -\frac{g_1^2}{2}\sum_{\alpha\in\Delta'}\alpha V'(\alpha\cdot\tilde{\omega}(\tilde{q})) - \frac{g_2^2}{2}\sum_{\alpha\in\Delta''}\alpha V'(\alpha\cdot\tilde{\omega}(\tilde{q})). \tag{4.20}$$

As discussed above a simple application of the map ω on both sides then yields (3.12). The non-trivial part of the reduction of the Lax pair is the cancellation of the unwanted terms, i.e. solving the functional equation (4.11). The root decomposition $\alpha = \beta + \gamma$ mixes elements in the two subsets Δ', Δ''. For instance, we find for $\alpha = \alpha_1$,

$$-\sum_{\beta\in\Delta}\varepsilon_{\alpha_1,\beta}\frac{g_\beta g_{\alpha_1+\beta}}{g_{\alpha_1}}V(\beta\cdot q) = g_2\{V(q_{13}) + V(q_{14}) - V(q_{23}) - V(q_{24})\} + g_2^2/g_1\{V(q_{15}) - V(q_{25})\}$$

with $q_{ij} = q_i - q_j$. Taking into account the reduced coordinates (4.16) one can solve (4.12), but finds upon inserting the solution into (4.11) for arbitrary roots α that one is forced to set

$$g_1 = g_2 = g. \tag{4.21}$$

That is, one has $g_\alpha = g$ for all crystallographic roots $\alpha \in \Delta$. With this restriction we recover after subtracting the superfluous vector (since $z'\cdot\alpha = 0$ for all roots α)

$$z' = \frac{e_1 + \cdots + e_5}{5}2\mathrm{i}g\sum_{j=1}^{5}\sum_{k=1,k\neq j}^{5}V(q_j - q_k) \tag{4.22}$$

from (4.12) the familiar solution of the A_ℓ series (see e.g. [29])

$$z_j = 2\mathrm{i}g\sum_{k=1,k\neq j}^{5}V(q_j - q_k), \tag{4.23}$$

albeit in our case the coordinates q_i have to be replaced by (4.16). Thus, we can conclude that the dynamical system defined through the Hamiltonian (1.2) allows for a Lax pair formulation and hence the quantities $I_k = \operatorname{Tr} L^k$ are conserved, i.e. $\mathrm{d}(\operatorname{Tr} L^k)/\mathrm{d}t = 0$. This is usually a strong indication that the model is indeed Liouville integrable. It remains to show that the aforementioned integrals of motion mutually Poisson commute and that at least $\tilde{\ell} = 2$ of them are non-vanishing and algebraically independent.

4.2. Integrals of motion

Having established the existence of the Lax operator L we may now use its explicit form

$$L = \mathrm{i}g\begin{pmatrix} p_1/\mathrm{i}g & x(q_{12}) & x(q_{13}) & x(q_{14}) & x(q_{15}) \\ x(q_{21}) & p_2/\mathrm{i}g & x(q_{23}) & x(q_{24}) & x(q_{25}) \\ x(q_{31}) & x(q_{32}) & p_3/\mathrm{i}g & x(q_{34}) & x(q_{35}) \\ x(q_{41}) & x(q_{42}) & x(q_{43}) & p_4/\mathrm{i}g & x(q_{45}) \\ x(q_{51}) & x(q_{52}) & x(q_{53}) & x(q_{54}) & p_5/\mathrm{i}g \end{pmatrix}, \tag{4.24}$$

to compute the integrals of motion. From examples of the crystallographic Calogero–Moser models it is known that the algebraically independent integrals of motion occur when the power k of $I_k = \mathrm{Tr}\, L^k$ matches the degrees $\{d_i\}_{i=1}^{\ell}$ of the Coxeter group. For A_4, the degrees are $d_i = 2, 3, 4, 5$, while in our case of interest, H_2, they are $\tilde{d}_i = 2, 5$ [13]. For instance, in the case of A_4 one verifies indeed that I_2, I_3, I_4, I_5 are algebraically independent, while for I_6 we have the relation

$$I_6 = \tfrac{3}{4} I_2 I_4 + \tfrac{1}{3} I_3^2 - \tfrac{1}{8} I_2^3, \tag{4.25}$$

where we have used the centre-of-mass constraints $\sum_i p_i = \sum_i q_i = 0$.

One might anticipate that due to the additional dependences in (4.16) some of the higher integrals of motion from the non-reduced A_4 theory must become algebraically dependent in the reduction. For instance, consider the integral of motion of degree 3 of the non-reduced model:

$$I_3(p,q) = \mathrm{Tr}\, L^3(p,q) = \sum_i p_i^3 + 3g^2 \sum_i p_i \sum_{j\neq i} V(q_i - q_j). \tag{4.26}$$

Since for the non-reduced as well as the reduced theories we have

$$\mathrm{Tr}\, L = p_1 + \cdots + p_5 = 0, \tag{4.27}$$

one might expect the reduced integral of motion of degree 3 $I_3^{\mathrm{red}} = I_3(\mu(p), \mu(q))$ to vanish. For purely algebraic reasons, we find the following simplification:

$$I_3^{\mathrm{red}} = 3g^2 \sum_i \mu(p)_i \sum_{j\neq i} V(\mu(q)_i - \mu(q)_j). \tag{4.28}$$

Because of the reduction (4.16) the purely kinetic term $\sum_i \mu(p)_i^3$ is zero. The remainder, however, does not vanish in general. The reason might be that the reduced set of crystallographic equations of motion (3.11) is more restrictive than (3.12) and that the non-vanishing of this integral of motion is a remnant of the reduction procedure. Consulting the literature we found that this issue is also not addressed in the context of folding. It certainly requires a deeper investigation of the model, for instance finding the explicit solutions of the equations of motion. In comparison, the non-crystallographic reduction of affine Toda field theories [28] involved two sets of complementary degrees, whose union gives again the degrees of A_4. We leave this question for future work.

In this context, it is also noteworthy that the match between the degrees of the Coxeter group and the powers of the integrals of motion, $k = d_i$, appears to be an observation based on examples rather than a rigorous mathematical theorem which applies to all known Calogero–Moser models. In particular, as its verification depends on the explicit form of a given Lax pair. For example, we followed for H_2 the Lax pair construction given in [24], which is based on the Coxeter group and the root system alone; see the appendix. One verifies in the case of the root-type representation, as stated in [24], that the Lax equation only holds for the rational potential; see our earlier remarks in the introduction. Explicit computation of the quantities $\mathrm{Tr}\, L^k$ shows that they vanish for $k = 1, 3, 5$. A similar analysis for the A_4 theory based on the root-type Lax pair gives an algebraically dependent expression for $\mathrm{Tr}\, L^3$. Thus, while one might expect the existence on an algebraically independent integral of motion at a certain degree d_i, not every Lax pair will provide one at the same power $k = d_i$.

5. Conclusions

The purpose of this paper has been to extend a recent reduction procedure [28] based on the embedding of non-crystallographic Coxeter groups into crystallographic ones to Calogero–Moser models. This leads us to propose new integrable systems based on $H_{2,3,4}$ by extending

from the known rational potential to the trigonometric, hyperbolic and elliptic cases. The new feature has been the appearance of an additional term in the potential energy, where the argument of the potential function is rescaled by the golden ratio. It is this extra term which is difficult to identify in the rational case, where it corresponds to a simple rescaling of the overall coupling constant. As we have discussed in the text the additional potential term can only be fully appreciated through the analysis of the underlying structure of the Coxeter groups.

While there are mathematical differences between the embeddings (1.4) and the folding [20] of a simply laced Lie algebra by a Dynkin diagram automorphism, we motivated our procedure by pointing out similarities and differences in the outcome. In particular, we showed that the reduced crystallographic Hamiltonian, which originally depends only on one coupling constant, still preserves invariance under the non-crystallographic Coxeter group if a second, independent coupling constant in front of the additional potential term is introduced. This is in close analogy with folding, where one also starts from a Hamiltonian incorporating only one coupling constant [31], while the folded Hamiltonian then allows for two independent couplings in front of the two Weyl group orbits corresponding to short and long roots [9–12].

We addressed the question of Liouville integrability of the new models by reducing the associated crystallographic Lax pair for the simplest case, namely H_2, and found that (1.2) indeed possesses higher conserved integrals of motion. We explained in detail where in the Lax pair construction a consistency requirement forces one to set the two coupling constants in front of the two potential terms equal. This does not preclude the possibility that another Lax pair construction might be possible which enables one to keep the two coupling constants independent, as it is the case with the models based on non-simply laced Lie algebras. The only basis on which to expect such an extension of the result found here is the Coxeter invariance of the Hamiltonian (3.4). While the latter does not necessarily imply Liouville integrability, it is the central, although not exclusive, criterion for exact solvability. However, the models (3.4) with $g_2 = 0$ have been previously discussed in the literature from both conceptual points of view, Lax pairs [24] and exact solvability [27], neither of these two approaches appears to have been successful beyond the rational potential[3]. This seems to indicate that the second potential term in (1.2) is indeed essential.

Despite the fact that we have focused for the Lax pair construction on the simplest model only, it should be evident from this example calculation that similar results are possible for the two other cases. In particular, the reduction procedure does not change any of the algebraic properties of the crystallographic Lax pairs. Moreover, our analysis of the mathematical structure associated with the embeddings (1.4) presented in section 2 of this paper allows one to accommodate also other formulations of Lax pairs [23, 24]; see the appendix. In this context, it would be interesting to verify whether also for these cases the two coupling constants in (3.9) have to be equal to ensure a consistent Lax pair.

To complete the proof of Liouville integrability one needs to verify that sufficiently many of the conserved quantities I_k are algebraically independent and Poisson commute. The latter step can be carried out using the concept of r-matrices [37, 40–44]. For the aforementioned reasons we expect that similar constructions based on the crystallographic r-matrices will carry through to the non-crystallographic models. What will be different is the number of algebraically independent integrals of motion, only half as many are needed due to the relation $\ell = 2\tilde{\ell}$ for the ranks of the two Coxeter groups in (1.4). That such a decrease in the number of

[3] In [39], a complete proof of Liouville integrability is presented for the classical Calogero–Moser models based on all root systems (including the non-crystallographic ones) and for all types of potentials. This proof makes use of the Lax pair formulation based on reflection operators of the Coxeter group in [24], which for the non-crystallographic groups only applies to the rational potential; see e.g. the comment after equation (4.43) and solution (4.44) to the functional equation.

independent integrals of motion occurs is to be expected from the change of variables (3.1) in the reduction procedure, which introduces additional dependences. We postpone these more involved questions to future work.

Starting from the results in this paper one can now proceed further and discuss the corresponding quantum-mechanical systems as well. In this context new aspects arise, such as invariant theory and Dunkl operators [45]. In the case of the rational quantum Calogero model it is customary to add a confining harmonic potential in order to obtain a discrete spectrum. Since we omitted this case from our discussion let us briefly mention that the reduction procedure applies then as well and by a computation along the same lines as that in section 3, one finds that the frequency of the harmonic oscillator is rescaled in the same manner as the coupling constant of the crystallographic Hamiltonian, compare with (3.5).

We conclude by emphasizing once more the general nature of the reduction from crystallographic to non-crystallographic Coxeter groups. Its possible applications are as widespread as that of the folding procedure [20]. The significant difference is that the non-crystallographic reduction not only yields an alternative description but achieves to maintain a connection with the theory of semi-simple Lie algebras whose rich mathematical structure has found applications in many physical areas beyond that discussed here. A prominent example where this connection might be relevant is the correspondence between Calogero–Moser models and supersymmetric Yang–Mills theory in four dimensions, see e.g. [46].

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Appendix. Comparison with the root-type Lax pair based on Coxeter groups

Our reduction of the Lie algebraic Lax pair for A_4 showed that the two coupling constants in (3.9) cannot be chosen independently. However, there is the possibility that other construction schemes not based on Lie algebras might be successful. If this would be the case we can set $g_2 = 0$ in (3.9) and obtain a non-crystallographic Hamiltonian (1.2) where only the first potential term is present. These types of models have been investigated in [24] by constructing Lax pairs based on representations of the Coxeter group. Let us briefly describe this alternative formulation in order to compare results. We shall concentrate on the 'root-type' representation; see [23, 24] for other possibilities. The following description applies to both the crystallographic and the non-crystallographic root systems.

Following [23, 24], we introduce a vector space V_Δ which is the linear span of the following set of basis vectors $\{|\alpha\rangle\}_{\alpha\in\Delta}$ labelled by the roots, i.e. V is the direct sum $V_\Delta = \bigoplus_{\alpha\in\Delta} V_\alpha$ with V_α being the one-dimensional space spanned by $|\alpha\rangle$. Obviously, the dimension of this space coincides with the number of roots ℓh, with h being the Coxeter number. The Lax pair will be represented on this space. To formulate the latter, one first introduces the following action of the Coxeter group:

$$w|\alpha\rangle := |w(\alpha)\rangle, \qquad \text{for all} \quad \alpha \in \Delta, \quad w \in \mathcal{W}. \tag{A.1}$$

As the root system is invariant under $\mathcal{W}$, so is V_Δ. With respect to this action the Weyl reflections $w = \sigma_\alpha$ assume the role of the step operators E_α in the previous Lie algebraic formulation of the Lax pair. To mimic the Cartan elements H_i one introduces the following

set of linear operators [23, 24]:

$$h_i|\alpha\rangle := \alpha^i|\alpha\rangle, \qquad \text{for all} \quad \alpha \in \Delta, \quad i = 1, \ldots, \ell. \tag{A.2}$$

They commute among themselves, $[h_i, h_j] = 0$, and satisfy the crucial relation [24]

$$[h_i, \sigma_\alpha] = \frac{2\alpha^i}{\alpha^2}(\alpha \cdot h)\sigma_\alpha \tag{A.3}$$

with the Weyl reflections. In addition one imposes a similar trace convention as for the Cartan generators, $\operatorname{Tr} h_i h_j = \text{const}\, \delta_{ij}$. In terms of these operators and the Weyl reflections the Lax pair now reads [24]

$$L = p \cdot h + \mathrm{i} \sum_{\alpha \in \Delta_+} g_\alpha x(\alpha \cdot q)(\alpha \cdot h)\sigma_\alpha \qquad \text{and} \qquad M = \mathrm{i} \sum_{\alpha \in \Delta_+} g_\alpha \frac{\alpha^2}{2} y(\alpha \cdot q)\sigma_\alpha. \tag{A.4}$$

Similar to the calculation in the Lie algebraic construction one shows that the Lax equation (4.9) is equivalent to the equations of motion provided $y = x'$ and certain unwanted terms cancel. The latter condition again leads to a functional equation which in the present construction reads [24]

$$\sum_{\alpha,\beta \in \Delta_R} g_\alpha g_\beta [\beta^2 x(\alpha \cdot q) y(\beta \cdot q)(\alpha \cdot \mu) - \alpha^2 y(\alpha \cdot q) x(\beta \cdot q)(\sigma_\alpha(\beta) \cdot \mu)] = 0. \tag{A.5}$$

Here the set Δ_R contains all pairs (α, β) of positive roots for which $R = \sigma_\alpha \sigma_\beta \in \mathcal{W}$ is a fixed rotation and $\mu = (\mu_1, \ldots, \mu_\ell)$ can be any vector. The advantage of this formulation is that it equally applies to crystallographic and non-crystallographic Coxeter groups. However, for non-crystallograpic systems the functional equation (A.5) is only satisfied for the rational potential and the Lax construction breaks down for the trigonometric, hyperbolic and elliptic cases [24].

A.1. The case H_2

This can be explicitly verified for the H_2 group by picking the following representation of the roots [13]:

$$\tilde{\beta}_k = \sqrt{2}\left(\cos\frac{\pi k}{5}, \sin\frac{\pi k}{5}\right), \qquad k = 1, 2, \ldots, 10.$$

In this representation, the root set for a fixed rotation $R = \tilde{\sigma}_{\tilde{\beta}_i}\tilde{\sigma}_{\tilde{\beta}_j}$ is given by [24]

$$\tilde{\Delta}_R = \{(\tilde{\beta}_{i+k}, \tilde{\beta}_{j+k}) : k = 0, 1, \ldots, 4\}.$$

One then verifies that the functional equation (A.5) does not hold beyond the rational potential. For the conserved charges, we find the expressions

$$\operatorname{Tr} L = \operatorname{Tr} L^3 = \operatorname{Tr} L^5 = 0$$

and

$$\frac{\operatorname{Tr} L^2}{10} = \frac{\operatorname{Tr} L^4}{15} = 2H = \tilde{p}_1^2 + \tilde{p}_2^2 + \frac{25g^2\left(\tilde{q}_1^2 + \tilde{q}_2^2\right)^4}{\left(\tilde{q}_1^5 - 10\tilde{q}_1^3\tilde{q}_2^2 + 5\tilde{q}_1\tilde{q}_2^4\right)^2}.$$

Note the absence of a non-trivial charge for degree 5.

A.2. The case A_4

Here we choose the same representation of the roots as in the context of the Lie algebraic Lax pair. The root-type representation is now 20-dimensional. Note that we have to shift the h_i-operators by a constant, $h_i \to h_i + 1/\sqrt{10}$, in order to ensure the aforementioned trace convention. This does not affect the commutation relations.

Now the root sets in the functional equation (A.5) involve at most three pairs of roots. An example is

$$\Delta_R = \{(\alpha_1, \alpha_3 + \alpha_4), (\alpha_1 + \alpha_3 + \alpha_4, \alpha_1), (\alpha_3 + \alpha_4, \alpha_1 + \alpha_3 + \alpha_4)\}, \qquad R = \sigma_{\alpha_1}\sigma_{\alpha_3+\alpha_4}.$$

Note the mixing of roots belonging to the subsets Δ', Δ'' as in the previous construction of the Lax pair based on the Lie algebra A_4. The functional equation (A.5) for the trigonometric case with $x(u) = \cot u$ reads explicitly

$$g_1(g_1 - g_2)\frac{\mu_1 \sin 2q_{12} - \mu_2 \sin 2q_{24} - 2\mu_4 \cos q_{14} \sin(q_{12} - q_{24})}{2\sin^2 q_{12} \sin^2 q_{24}} = 0.$$

From this and similar equations, we again infer that we need to set $g_1 = g_2$ to satisfy the Lax equation.

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