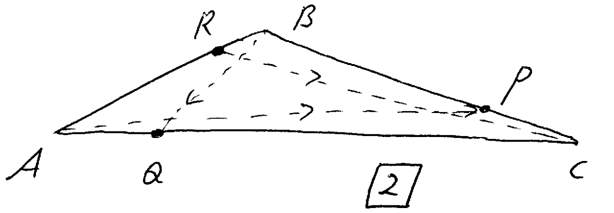
Geometry & Vectors (Exam 05)

Solutions and marking scheme:

1)



$\boxed{2}$

$\overline{BP} = \frac{3}{4}\overline{BC} \Rightarrow \overline{PC} = \frac{1}{4}\overline{BC}$

$\overline{CQ} = \frac{3}{4}\overline{CA} \Rightarrow \overline{AQ} = \frac{1}{4}\overline{AC}$

$\overline{AR} = \frac{3}{4}\overline{AB} \Rightarrow \overline{RB} = \frac{1}{4}\overline{AB}$

$$\Rightarrow \overrightarrow{AP} = \overrightarrow{AB} + \overrightarrow{BP} = \overrightarrow{AB} + \frac{3}{4}\overrightarrow{BC} \qquad \boxed{6}$$

$$\overrightarrow{BQ} = -\overrightarrow{AB} + \overrightarrow{AQ} = -\overrightarrow{AB} + \frac{1}{4}\overrightarrow{AC} = -\overrightarrow{AB} + \frac{1}{4}(\overrightarrow{AB} + \overrightarrow{BC})$$

$$\overrightarrow{CR} = -\overrightarrow{BC} + \overrightarrow{BR} = -\overrightarrow{BC} - \frac{1}{4}\overrightarrow{AB}$$

$$\Rightarrow \overrightarrow{AP} + \overrightarrow{BQ} + \overrightarrow{CR} = \cancel{\overrightarrow{AB}} + \frac{3}{4}\overrightarrow{BC} - \cancel{\overrightarrow{AB}} + \underline{\frac{1}{4}\overrightarrow{AB}} + \frac{1}{4}\overrightarrow{BC} - \overrightarrow{BC} \underline{- \frac{1}{4}\overrightarrow{A}}$$

$$= 0 \qquad 8$$

2)

$$\vec{u} = 2\vec{i} - \vec{j} + \vec{k} \qquad \vec{v} = 3\vec{i} - 3\vec{j}$$

i) $\vec{u}\cdot\vec{v} = 6 + 3 = 9$

$|\vec{v}| = \sqrt{3\cdot 3 + 3\cdot 3} = 3\sqrt{2}$

$|\vec{u}| = \sqrt{2\cdot 2 + 1 + 1} = \sqrt{6}$

$$\Rightarrow \cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{9}{3\cdot\sqrt{2}\sqrt{6}} = \frac{\sqrt{3}}{2} \Rightarrow \underline{\underline{\theta = \frac{\pi}{6}}} \qquad \boxed{3}$$

ii) general vector of the form $\vec{w} = a\vec{i} + b\vec{j} + c\vec{k}$

unit vector: $|\vec{w}| = 1 = a^2 + b^2 + c^2$

$\vec{v} \perp \vec{w}$: $\vec{v}\cdot\vec{w} = 0 = 3a - 3b \Rightarrow \underline{a = b}$

$\vec{u} \perp \vec{w}$: $\vec{u}\cdot\vec{w} = 0 = 2a - b + c \Rightarrow \underline{a = -c}$

$\Rightarrow 3a^2 = 1 \Rightarrow \underline{a = \pm\frac{1}{\sqrt{3}}}$

$$\Rightarrow \underline{\underline{\vec{W}_{1/2} = \pm\frac{1}{\sqrt{3}}(\vec{i} + \vec{j} - \vec{k})}} \qquad \boxed{3}$$

iii) triangle identity: $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$

$\vec{u} + \vec{v} = 5\vec{i} - 4\vec{j} + \vec{k}$ $\qquad \boxed{2}$

$|\vec{u} + \vec{v}|^2 = 25 + 16 + 1 = 42$

$|\vec{u}| = \sqrt{6}$

$|\vec{v}| = 3\sqrt{2}$

$$\Rightarrow \sqrt{42} \leq \sqrt{6} + 3\sqrt{2}$$

$$\Leftrightarrow 6.48\ldots \leq 6.69\ldots \quad ✓ \qquad 8$$

3)

$$(\vec{u} \times \vec{v}) \cdot (\vec{w} \times \vec{x}) = (\vec{u} \cdot \vec{w})(\vec{v} \cdot \vec{x}) - (\vec{u} \cdot \vec{x})(\vec{v} \cdot \vec{w})$$

$$\vec{u} = \vec{j} + 2k \qquad \vec{v} = -\vec{i} + 3\vec{j} \qquad \vec{w} = \vec{i} - \vec{j} \qquad \vec{x} = \vec{i} + \vec{j} - \vec{k}$$

RHS:

$$\left.\begin{array}{l} \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 2 \\ -1 & 3 & 0 \end{vmatrix} = -6\vec{i} - 2\vec{j} + \vec{k} \\ \vec{w} \times \vec{x} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{vmatrix} = \vec{i} + \vec{j} + 2\vec{k} \end{array}\right\} \Rightarrow (\vec{u} \times \vec{v}) \cdot (\vec{w} \times \vec{x}) = -6 - 2 + 2 = -$$

$\boxed{4}$

LHS:

$$\vec{u} \cdot \vec{w} = -1 \qquad \vec{u} \cdot \vec{x} = 1 - 2 = -1 \qquad \vec{v} \cdot \vec{x} = -1 + 3 = 2 \qquad \vec{v} \cdot \vec{w} = -$$

$$\Rightarrow (\vec{u} \cdot \vec{w})(\vec{v} \cdot \vec{x}) - (\vec{u} \cdot \vec{x})(\vec{v} \cdot \vec{w}) = (-1)(2) - (-1)(-4) = -2 - 4 = \underline{\underline{-6}} \quad \boxed{4}$$

$$\Rightarrow \quad LHS = RHS$$

8

4)

$$2x^2 + 3y^2 - 4x + 5y + 4 = 0$$

Complete the square:

$$2(x^2 - 2x + 1) - 2 + 3\left(y^2 + \frac{5}{3}y + \left(\frac{5}{6}\right)^2\right) - \left(\frac{5}{6}\right)^2 3 + 4 = 0$$

$$\Leftrightarrow \qquad 2(x-1)^2 + 3\left(y + \frac{5}{6}\right)^2 = 2 - 4 + \frac{25}{12} = \frac{1}{12}$$

$\Rightarrow$ the normal form of the ellipse is

$\boxed{2}$
$$\frac{(x-1)^2}{1/24} + \frac{\left(y + \frac{5}{6}\right)^2}{1/36} = 1 \quad \Rightarrow \quad a^2 = \frac{1}{24}, \quad b^2 = \frac{1}{36}$$

$\boxed{1}$ $\Rightarrow$ the centre is at $\underline{\underline{\left(1, -\frac{5}{6}\right)}}$

$\boxed{1}$ $\Rightarrow$ the length of the major axis is $2a = 2/\sqrt{24} = \underline{\underline{\frac{1}{\sqrt{6}}}}$

$\Rightarrow$ the length of the minor axis is $2b = 2/\sqrt{36} = \underline{\underline{\frac{1}{3}}}$

$\boxed{1}$ $\Rightarrow$ the eccentricity is $e = \sqrt{1 - b^2/a^2} = \sqrt{1 - \frac{24}{36}} = \sqrt{1 - \frac{2}{3}} = \frac{1}{\sqrt{\ }}$

8

$\boxed{1}$ $\Rightarrow$ the foci are at $(1 \pm ea, -\frac{5}{6}) = \underline{\underline{\left(1 \pm \frac{1}{6\sqrt{2}}, -\frac{5}{6}\right)}}$

$\boxed{1}$ $\Rightarrow$ the vertices are at $(1 \pm a, -\frac{5}{6}) = \underline{\underline{\left(1 \pm \frac{1}{2\sqrt{6}}, -\frac{5}{6}\right)}}$

$\boxed{1}$ $\Rightarrow$ the equation of the directrix is $x = 1 - \frac{b^2}{ae} = 1 - \frac{6\sqrt{2}}{36} = 1 -$

5) i) Take $P(x, y, z)$ to be an arbitrary point in the plane.

The vectors $\vec{AB}$, $\vec{AC}$ and $\vec{CP}$ are in the plane.

$$\vec{AB} = \vec{OB} - \vec{OA} = 2\vec{i} - 3\vec{j} + 5\vec{k}$$

$$\vec{AC} = \vec{OC} - \vec{OA} = 4\vec{i} - 2\vec{j} + 3\vec{k}$$

$$\vec{CP} = (x-11)\vec{i} + (y+1)\vec{j} + (z-5)\vec{k}$$

The vector $\vec{AB} \times \vec{AC}$ is perpendicular to the plane.

$\Rightarrow \quad \vec{CP} \cdot (\vec{AB} \times \vec{AC}) = 0$

$$\Leftrightarrow \quad \begin{vmatrix} (x-11) & (y+1) & (z-5) \\ 2 & -3 & 5 \\ 4 & -2 & 3 \end{vmatrix} = 0$$

$\boxed{6}$

$\Leftrightarrow \quad (x-11)(-9+10) - (y+1)(6-20) + (z-5)(-4+12) = 0$

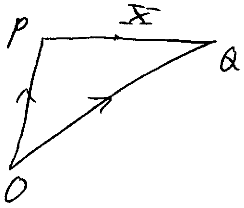
$\Leftrightarrow \quad \underline{\underline{x + 14y + 8z - 37 = 0 = f_{plane}(x, y, z)}}$

ii) The distance of a point $P(x_0, y_0, z_0)$ from a plane described by $ax + by + cz + d = 0$ is $d = \left| \dfrac{a x_0 + b y_0 + c z_0 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$ $\boxed{2}$

Distance P to the plane: $d = \left| \dfrac{16 + 15 \cdot 14 + 9 \cdot 8 - 37}{\sqrt{1 + 196 + 64}} \right| = \dfrac{261}{\sqrt{261}} = 3\sqrt{29}$

(8)

6)



Any point $X(x, y, z)$ on $\overleftrightarrow{PQ}$ is described by

$$\vec{OX} = \vec{OP} + \lambda \vec{PQ}$$

$\Rightarrow \quad x\vec{i} + y\vec{j} + z\vec{k} = (-5\vec{i} + 4\vec{j} + \vec{k}) + \lambda(6\vec{i} - \vec{j} + \vec{k})$

$\Rightarrow \quad X(x, y, z) = X(-5 + 6\lambda, 4 - \lambda, 1 + \lambda)$ $\boxed{4}$

$\Rightarrow$ point of intersection

$$3(6\lambda - 5) - (4 - \lambda) + 6(1 + \lambda) - 87 = 0$$

(8)

$\Leftrightarrow \quad 25\lambda - 100 = 0 \quad \Rightarrow \underline{\lambda = 4}$ $\boxed{4}$

$\Rightarrow$ point of intersection $P_I(-5 + 6 \cdot 4, 4 - 4, 1 + 4) = \underline{P_I(19, 0, 5)}$

7) i) Axiom 1 (line axiom):

[3] Through any two distinct points P and Q there is exactly one line $L = \overleftrightarrow{PQ}$.

Axiom 2 (plane axiom):

[3] Through any three noncollinear points there is exactly one plane.

Axiom 3 (dimension axiom):

[3] Any line contains at least two distinct points.
Any plane contains at least two distinct lines.
There are at least 2 distinct planes in space.

Axiom 4 (line-plane intersection axiom):

[3] If two distinct points P_1, P_2 on a line L lie in a plane $\mathcal{P}$, then the whole line lies in $\mathcal{P}$.
$(P_1, P_2 \in L, \mathcal{P} \Rightarrow L \in \mathcal{P})$

Axiom 5 (parallel axiom):

[3] For a given point P and line L there is one and only one line L' which passes through P and is parallel to L.

ii) Proof: (seen)

- from axiom 3 $\Rightarrow \exists$ at least two distinct points $Q, R \in$
- [11] P, Q, R are not collinear since the unique line L through Q and R does not pass through P
- (26) from axiom 2 $\Rightarrow$ there is a unique plane $\mathcal{P}$ through $P, Q,$
- $\mathcal{P}$ contains two points of L, namely $Q, R \Rightarrow$ from axiom
$L \in \mathcal{P}$ q.e.

8) i) $\vec{XU}\cdot\vec{VW} + \vec{XV}\cdot\vec{WU} + \vec{XW}\cdot\vec{UV} = 0$

$$(\vec u - \vec x)\cdot(\vec w - \vec v) + (\vec v - \vec x)\cdot(\vec u - \vec w) + (\vec w - \vec x)\cdot(\vec v - \vec u) = 0$$

$\boxed{4}$ $\vec u\cdot\vec w - \vec u\cdot\vec v - \vec x\cdot\vec w + \cancel{\vec x\cdot\vec v} + \vec v\cdot\vec u - \vec v\cdot\vec w - \vec x\cdot\vec u + \vec x\cdot\vec w + \vec w\cdot\vec v - \vec w\cdot\vec u - \cancel{\vec x}$

$+ \vec x\cdot\vec u = 0$

ii) $|\vec{UV}|^2 - |\vec{VW}|^2 + |\vec{WX}|^2 - |\vec{XU}|^2$

$$= \cancel{|\vec u|^2} + \cancel{|\vec v|^2} - 2\vec u\cdot\vec v - \cancel{|\vec v|^2} - \cancel{|\vec w|^2} + 2\vec v\cdot\vec w + \cancel{|\vec w|^2} + \cancel{|\vec x|^2} - 2\vec w\cdot\vec x$$
$$- \cancel{|\vec x|^2} - \cancel{|\vec u|^2} + 2\vec x\cdot\vec u$$

$\boxed{4}$

$$= 2\vec v\cdot(\vec w - \vec u) + 2\vec x\cdot(\vec u - \vec w)$$

$$= 2(\vec w - \vec u)\cdot(\vec v - \vec x) = 2\,\vec{UW}\cdot\vec{XV}$$

iii) $\vec u = \gamma\vec i, \quad \vec v = v_1\vec i + v_2\vec j + v_3\vec k, \quad \vec w = w_1\vec i + w_2\vec j + w_3\vec k$

$$\vec u \times \vec v \times \vec w = \vec u \times \begin{vmatrix} \vec i & \vec j & \vec k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= \vec u \times \left[\underbrace{(v_2w_3 - v_3w_2)}_{a}\vec i + \underbrace{(v_3w_1 - v_1w_3)}_{b}\vec j + \underbrace{(v_1w_2 - v_2w_1)}_{c}\vec k\right]$$

$\boxed{14}$

$$= \begin{vmatrix} \vec i & \vec j & \vec k \\ \gamma & 0 & 0 \\ a & b & c \end{vmatrix} = -\gamma c\vec j + \gamma b\vec k$$

$$= (-\gamma v_1w_2 + \gamma v_2w_1)\vec j + \gamma(v_3w_1 - \gamma v_1w_3)\vec k$$

$$= \gamma w_1(v_2\vec j + v_3\vec k) - \gamma v_1(w_2\vec j + w_3\vec k) + \underbrace{\gamma v_1w_1\vec i - \gamma v_1 w}_{+0}$$

$$= \underbrace{\gamma w_1}_{\vec u\cdot\vec w}\underbrace{(v_1\vec i + v_2\vec j + v_3\vec k)}_{\vec v} - \underbrace{\gamma v_1}_{\vec u\cdot\vec v}\underbrace{(w_1\vec i + w_2\vec j + w_3\vec k)}_{\vec w}$$

iv) $(\vec u\times\vec v)\times\vec w\times\vec x = \underbrace{[(\vec u\times\vec v)\cdot\vec x]}_{\vec u\cdot(\vec v\times\vec x)}\vec w - \underbrace{[(\vec u\times\vec v)\cdot\vec w]}_{\vec u\cdot(\vec v\times\vec w)}\vec x$

$\boxed{4}$

26

9) i) $Y = m_1 X + c_1 \qquad Y = m_2 x + c_2$

angle: $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2} \qquad m_1 = \dfrac{1}{\sqrt{3}}, \quad m_2 = -\dfrac{1}{\sqrt{3}}$

[4]

$\Rightarrow \tan\theta = \dfrac{1/\sqrt{3} + 1/\sqrt{3}}{1 - \frac{1}{3}} = \dfrac{2/\sqrt{3}}{2/3} = \dfrac{3}{\sqrt{3}} = \sqrt{3} \Rightarrow \underline{\underline{\theta = \frac{\pi}{3}}}$

ii) For a hyperbola in the normal form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

the asymptotes are $aY - bx = 0$ and $aY + bx = 0$

[9] Compare with $Y - \frac{1}{\sqrt{3}}x = 0$ and $Y + \frac{1}{\sqrt{3}}x = 0 \Rightarrow \underline{\frac{b}{a} = \frac{1}{\sqrt{3}}}$

From $f(r,\theta) = 1 - \dfrac{k}{1 - e\cos\theta} \qquad \Rightarrow \underline{k} = -\sqrt{3} = \underline{-\dfrac{b^2}{a}}$

$\Rightarrow \underline{b = 3}$ and $\underline{a = 3\sqrt{3}} \qquad \Rightarrow \underline{\underline{\dfrac{x^2}{27} - \dfrac{y^2}{9} = 1}}$

iii) The plane contains the vectors

$\vec{u} = 3\vec{i} + 2\vec{j} + \vec{k}$ and $\vec{v} = 3\vec{i} + \vec{j} - 4\vec{k}$

Any point on L_3 can be parameterised as

[9]

$$P_\lambda(5 + 3\lambda, 1 + 2\lambda, -1 + \lambda) \in L_3$$

For any point $P(x, Y, z) \in$ the plane $\overrightarrow{P_0P}$ is in the plane, such that $\overrightarrow{P_0P} \perp \vec{u} \times \vec{v} \Leftrightarrow \overrightarrow{P_0P} \cdot (\vec{u} \times \vec{v}) = 0$

$$\Rightarrow \begin{vmatrix} (x-5) & (Y-1) & (z+1) \\ 3 & 2 & 1 \\ 3 & 1 & -4 \end{vmatrix} = (x-5)(-8-1) - (Y-1)(-12-3) + (z+1)(3-6) =$$

$$= \underline{\underline{-9x + 15Y - 3z + 27 = 0}}$$

iv)

$P_\mu(-1 + 2\mu, 2 + 3\mu, -2 + \mu) = P_\lambda = P$ for intersection

[4]

$\Rightarrow (1)\ 5 + 3\lambda = -1 + 2\mu \qquad (3) \Rightarrow \mu = \lambda + 1 \qquad \Rightarrow P_{\mu=-3} = P_{\lambda=-4}$

(26)

$(2)\ 1 + 2\lambda = 2 + 3\mu \qquad (2) \Rightarrow 1 + 2\lambda = 2 + 3 + 3\lambda$

$(3)\ \lambda - 1 = \mu - 2 \qquad \Rightarrow \underline{\lambda = -4} \Rightarrow \underline{\mu = -3} \Rightarrow \underline{\underline{P(-7, -7, -5)}}$

(1) also holds