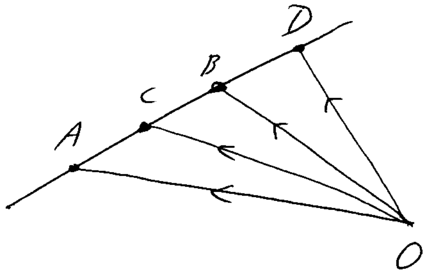
# Geometry & Vectors (2007)

1) i) $\vec{OA} = \frac{3}{2}\vec{i} + \vec{j}$ $\qquad \vec{OB} = 3\vec{i} + 4\vec{j}$




$AC : CB = 2 : 1 \quad \Rightarrow \quad \vec{AC} = 2\vec{CB}$

$\Leftrightarrow \vec{OC} - \vec{OA} = 2\vec{OB} - 2\vec{OC}$

$\Rightarrow \quad 3\vec{OC} = 2\vec{OB} + \vec{OA}$

$= 6\vec{i} + 8\vec{j} + \frac{3}{2}\vec{i} + \vec{j}$

$\Rightarrow \vec{OC} = \frac{5}{2}\vec{i} + 3\vec{j}$ (4)

ii) Two points X and Y on a line L are said to divide AB harmonically if the ratios AX : XB and AY : YB are the same except for a sign, i.e. there exist two scalars (1) $\lambda$ and $\mu$ such that $AX : XB = \lambda : \mu$ and $AY : YB = -\lambda :$

iii) From ii) : $\quad \vec{AC} = \frac{\lambda}{\mu}\vec{CB} \qquad \vec{AD} = -\frac{\lambda}{\mu}\vec{DB} = \frac{\lambda}{\mu}\vec{BD}$

$$\left.\begin{aligned} \vec{AC} &= -\vec{OA} + \vec{OC} = -\frac{3}{2}\vec{i} - \vec{j} + \frac{5}{2}\vec{i} + 3\vec{j} = \vec{i} + 2\vec{j} \\ \vec{CB} &= -\vec{OC} + \vec{OB} = -\frac{5}{2}\vec{i} - 3\vec{j} + 3\vec{i} + 4\vec{j} = \frac{1}{2}\vec{i} + \vec{j} \end{aligned}\right\} \Rightarrow \vec{i} + 2\vec{j} = \frac{\lambda}{\mu}\left(\frac{1}{2}\vec{i} + \vec{j}\right)$$

$\Rightarrow \lambda/\mu = 2$

$\vec{AD} = \vec{AB} + \vec{BD} = \vec{AB} + \frac{\mu}{\lambda}\vec{AD} \quad \Rightarrow \vec{AD} = 2\vec{AB} = -\vec{OA} + \vec{OD}$ (5)

$\Rightarrow \vec{OD} = 2\vec{AB} + \vec{OA} = -2\vec{OA} + 2\vec{OB} + \vec{OA}$

$\Rightarrow \vec{OD} = 2\vec{OB} - \vec{OA} = 6\vec{i} + 8\vec{j} - \frac{3}{2}\vec{i} - \vec{j} = \frac{9}{2}\vec{i} + 7\vec{j}$

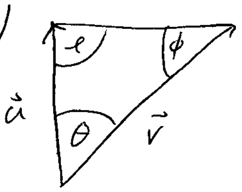
$\boxed{\Sigma = 10}$

2) i) $|\vec{u}|^2 = \vec{u}\cdot\vec{u} = \frac{1}{4}\left(T_+^2 + 1 + T_-^2\right) = \frac{1}{4}\left(1 + T_+ + 1 + 1 - T_-\right) = 1$

$|\vec{v}|^2 = \vec{v}\cdot\vec{v} = \frac{1}{4}\left(T_+^2 + 1 + T_-^2\right) = 1$

$|\vec{w}|^2 = \vec{w}\cdot\vec{w} = \frac{1}{4}\left(1 + T_-^2 + T_+^2\right) = 1$ (2)

ii)




$\cos\theta = \frac{\vec{v}\cdot\vec{u}}{|\vec{v}||\vec{u}|} = \vec{v}\cdot\vec{u} = \frac{1}{4}\left(-T_+^2 - 1 + T_-^2\right)$

$= \frac{1}{4}\left(-T_+ - 1 - 1 + 1 - T_-\right) = \frac{1}{4}\left(-1 - (T_+ + T_-)\right) = -\frac{1}{4}\left(\sqrt{5} + 1\right) = -$

$\Rightarrow \theta = \frac{4\pi}{5}$

$$\cos\varphi = \frac{-\vec{u}\cdot(\vec{v}-\vec{u})}{|\vec{u}|\,|\vec{v}-\vec{u}|} = \frac{-\vec{u}\cdot\vec{v}+|\vec{u}|^2}{|\vec{u}|\,|\vec{v}-\vec{u}|} = \frac{1-\vec{u}\cdot\vec{v}}{|\vec{v}-\vec{u}|}$$

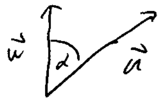
$$\cos\psi = \frac{-\vec{v}\cdot(\vec{u}-\vec{v})}{|\vec{v}|\,|\vec{v}-\vec{u}|} = \frac{-\vec{v}\cdot\vec{u}+|\vec{v}|^2}{|\vec{v}|\,|\vec{u}-\vec{v}|} = \frac{1-\vec{u}\cdot\vec{v}}{|\vec{v}-\vec{u}|}$$

$$\Rightarrow \underline{\varphi = \psi}$$

(5)

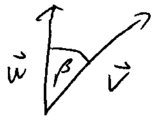
$$\Rightarrow 2\varphi + \frac{4\pi}{5} = \pi \Rightarrow \underline{\varphi = \psi = \frac{\pi}{10}}$$

iii)




$$\cos\alpha = \frac{\vec{w}\cdot\vec{u}}{|\vec{u}|\,|\vec{w}|} = \vec{w}\cdot\vec{u} = \frac{1}{4}\left(T_+ - T_- - T_+T_-\right) = \frac{1}{4}(1-1) = 0$$

$$\Rightarrow \underline{\alpha = \frac{\pi}{2}}$$




$$\cos\beta = \frac{\vec{w}\cdot\vec{v}}{|\vec{w}|\,|\vec{v}|} = \vec{w}\cdot\vec{v} = \frac{1}{4}\left(-T_+ + T_- - T_+T_-\right) = -\frac{1}{2}$$

(3)

$$\Rightarrow \underline{\beta = \frac{2\pi}{3}}$$

$\boxed{\Sigma = 10}$

3) i) Experiment 1: $\vec{B} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

$$q(\vec{v}\times\vec{B}) = q\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 1 & 0\\ b_1 & b_2 & b_3\end{vmatrix} = \vec{F} = q\left(-2\vec{i}+4\vec{j}-5\vec{k}\right)$$

$$\Rightarrow \vec{i}\,b_3 - 2b_3\vec{j} + (2b_2-b_1)\vec{k} = -2\vec{i}+4\vec{j}-5\vec{k}$$

$$\Rightarrow \underline{b_3 = -2} \qquad 2b_2 - b_1 = -5 \Rightarrow \underline{b_1 = 2b_2+5}$$

Experiment 2:

$$\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & -2 & 5\\ b_1 & b_2 & -2\end{vmatrix} = (4-5b_2)\vec{i} + (6+5b_1)\vec{j} + (2b_1+3b_2)\vec{k}$$

$$= 19\vec{i} + \vec{j} - 11\vec{k} \Rightarrow \underline{b_2 = -3} \qquad \underline{b_1 = -1}$$

$$\Rightarrow \underline{\vec{B} = -\vec{i} - 3\vec{j} - 2\vec{k}}$$

(3)

ii) Exp. 3:

$$\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ v_1 & v_2 & v_3\\ -1 & -3 & -2\end{vmatrix} = \left(-2v_2+3v_3\right)\vec{i} + \left(2v_1 - v_3\right)\vec{j} + \left(v_2 - 3v_1\,\vec{k}\right)$$

$$= 6\vec{i} + 2\vec{j} - 6\vec{k}$$

$\Rightarrow \quad -2V_2 + 3V_3 = 6 \qquad \Rightarrow \quad V_2 = -3 + \frac{3}{2}V_3$

$2V_1 - V_3 = 2 \qquad \Rightarrow \quad V_1 = 1 + V_3/2$

$V_2 - 3V_1 = -6 \qquad \Rightarrow \quad -3 + \frac{3}{2}V_3 - 3 - \frac{3}{2}V_3 = -6 \quad \forall\, V_3$

The velocity $\vec{V} = \vec{i} - 3\vec{j}$ is possible for $V_3 = 0$

The velocity $\vec{V} = 2\vec{i} + 2\vec{k}$ is obtained for $V_3 = 2$, $\therefore$ the direction $\vec{i} + \vec{k}$ is possible.

2

(iii) Suppose there is such a direction, then we should be able to solve

$$q\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ V_1 & V_2 & V_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{F} = q(f_1\vec{i} + f_2\vec{j} + f_3\vec{k})$$

for $V_1, V_2, V_3$.

$$\Rightarrow \vec{i}(V_2 b_3 - b_2 V_3) + \vec{j}(b_1 V_3 - V_1 b_3) + \vec{k}(V_1 b_2 - V_2 b_1) = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$$

$$\Rightarrow \quad \begin{aligned} V_2 b_3 - b_2 V_3 &= f_1 \\ V_3 b_1 - V_1 b_3 &= f_2 \\ V_1 b_2 - V_2 b_1 &= f_3 \end{aligned} \quad \Leftrightarrow \quad \underbrace{\begin{pmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{pmatrix}}_{M} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

$\because \det M = 0 \quad \Rightarrow M^{-1}$ does not exist

$\Rightarrow$ No solution for $V_1 V_2 V_3$ which is unique

5

$\Rightarrow$ One needs at least two expiments.

$\boxed{\Sigma = 10}$

4) $5x^2 + 9y^2 = 180 \quad \Rightarrow \quad \dfrac{x^2}{36} + \dfrac{y^2}{20} = 1$

For $Q(x_0, y_0)$ common to the ellipse and the tangent, the equation for the tangent is

$$\mathcal{L}: \quad \frac{x x_0}{36} + \frac{y y_0}{20} = 1$$

$P(9,0) \in \mathcal{L} \Rightarrow \frac{9x_0}{36} = 1 \Rightarrow \underline{x_0 = 4}$

$Q(x_0, y_0) \in \text{ellipse} \Rightarrow \frac{x_0^2}{36} + \frac{y_0^2}{20} = 1 \Rightarrow \frac{16}{36} + \frac{y_0^2}{20} = 1$

$\Rightarrow \frac{y_0^2}{20} = \frac{5}{9} \Rightarrow \underline{y_0 = \pm \frac{1}{3}}$

$\Rightarrow \mathcal{L}_{1/2}: \quad \frac{x \, 4}{36} \pm \frac{y}{20} \frac{10}{3} = 1$

$\Leftrightarrow \quad \frac{x}{9} \pm \frac{y}{6} = 1 \Rightarrow \underline{\underline{y = \pm\left(\frac{2}{3}x - 6\right)}}$ (10)

The gradients are $\underline{\underline{m = \pm \frac{2}{3}}}$.

$\boxed{\Sigma = 10}$

5) i) $A(0,4,-3) \quad B(1,2,1) \Rightarrow \vec{AB} = \vec{i} - 2\vec{j} + 4\vec{k}$

$\Rightarrow$ equation of the line through A and B

$\mathcal{L}: \quad \frac{x}{1} = \frac{y-4}{-2} = \frac{z+3}{4} = \lambda$

$\Rightarrow P(\lambda, -2\lambda+4, 4\lambda-3) \in \mathcal{L}$

$\Rightarrow P \in xz\text{-plane for } \lambda = 2$

$\Rightarrow$ The point of intersection is $\underline{\underline{P(2,0,5)}}$ (5)

ii) $\mathcal{L}$ intersects $\mathcal{P}$ for

$5\lambda - 3(4-2\lambda) + 2(4\lambda-3) = 1$

$5\lambda - 12 + 6\lambda + 8\lambda - 6 - 1 = 0$

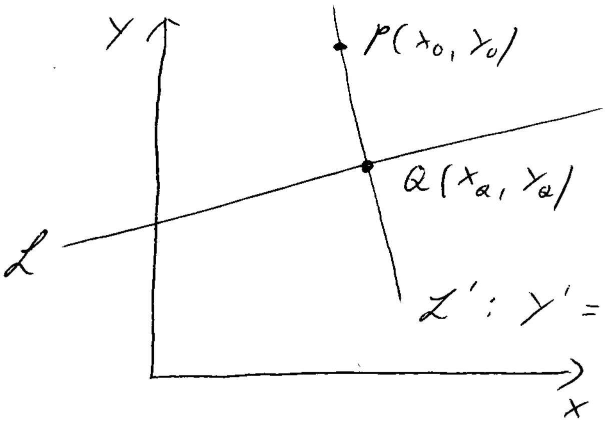
$19\lambda - 19 = 0 \Rightarrow \underline{\lambda = 1}$

(5)

$\Rightarrow \underline{\underline{P(1,2,1) = \mathcal{P} \cap \mathcal{L}}}$

$\boxed{\Sigma = 10}$

6/ i) (seen)




$$y = -\frac{\alpha}{\beta}x - \frac{\gamma}{\beta} = mx + k$$

$$\underline{m = -\alpha/\beta}$$

$$\underline{k = -\gamma/\beta}$$

The distance between P and Q is

$$d^2 = (y_Q - y_0)^2 + (x_Q - x_0)^2 \qquad (0)$$

$\mathcal{L} \perp \mathcal{L}' \Rightarrow \underline{m' = -\frac{1}{m}}$

$P \in \mathcal{L}' \Rightarrow y_0 = -\frac{1}{m}x_0 + k' \Rightarrow \underline{k' = y_0 + \frac{1}{m}x_0}$

$Q \in \mathcal{L} \Rightarrow y_Q = m x_Q + k \qquad (1)$

$Q \in \mathcal{L}' \Rightarrow y_Q = -\frac{1}{m}x_Q + k' \qquad (2)$

$(1)-(2)$ : $0 = \left(m + \frac{1}{m}\right)x_Q + (k - k') \Rightarrow \underline{x_Q = \frac{m(k'-k)}{1+m^2}}$

into (2) : $\underline{y_Q = \frac{k-k'}{1+m^2} + k'}$

(0) : $d^2 = \Big(\frac{k-k'}{1+m^2} + \underbrace{k' - y_0}_{x_0/m}\Big)^2 + \left(\frac{m(k'-k)}{1+m^2} - x_0\right)^2$

$$= (1+m^2)\left(\frac{k-k'}{1+m^2} + \frac{x_0}{m}\right)^2$$

$$= \frac{1}{1+m^2}\left(k - k' + \frac{1+m^2}{m}x_0\right)^2$$

$$= \frac{1}{1+m^2}\left(k - y_0 - \frac{1}{m}x_0 + \frac{1}{m}x_0 + m x_0\right)^2$$

$$= \frac{1}{1+\alpha^2/\beta^2}\left(-\frac{\gamma}{\beta} - y_0 - \frac{\alpha}{\beta}x_0\right)^2$$

$$\underline{d = \left|\frac{\alpha x_0 + \beta y_0 + \gamma}{\sqrt{\alpha^2+\beta^2}}\right|} \Leftarrow \quad = \frac{1}{\alpha^2+\beta^2}\left(-\gamma - \beta y_0 - \alpha x_0\right)^2$$

19

ii) $L_1: \quad 2y - 5x - 6 = 0 \quad \Rightarrow \quad y = \frac{5}{2}x + 3$

$L_2: \quad 4y - 10x + 8 = 0 \quad \Rightarrow \quad y = \frac{5}{2}x - 2$

$m_1 = m_2 \Rightarrow$ the two lines are parallel

By Euclid's $5^{th}$ axiom: "For a given point $P$ and line $L$ there is one and only one line $L'$ which passes through $P$ and is parallel to $L$" we can just take any point on $P \in L_1$ and compute the distance to $L_2$. (5)

For instance $P(0,3) \in L_1$

$$\Rightarrow \underline{\underline{d}} = \left| \frac{-10\cdot 0 + 4\cdot 3 + 8}{\sqrt{100+16}} \right| = \left| \frac{20}{\sqrt{116}} \right| = \underline{\underline{\frac{10}{\sqrt{29}}}} \quad (1)$$

$\boxed{\Sigma = 25}$

7) i) $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent if we can find $\lambda_1, \lambda_2, \lambda_3 \neq 0$ such that $\lambda_1 \vec{u} + \lambda_2 \vec{v} + \lambda_3 \vec{w} = 0$

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \quad \text{for} \quad \begin{aligned} \vec{u} &= u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k} \\ \vec{v} &= v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k} \\ \vec{w} &= w_1 \vec{i} + w_2 \vec{j} + w_3 \vec{k} \end{aligned}$$

The determinant is zero iff some rows or columns are multiples of others $\Leftrightarrow \vec{u} \cdot (\vec{v} \times \vec{w}) = 0$
$\Leftrightarrow \vec{u}, \vec{v}, \vec{w}$ are linearly dependent (5)

ii) $$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 1 & -3 & -2 \\ 3 & -\gamma & 1 \\ 2 & 0 & \gamma \end{vmatrix} = 2 \begin{vmatrix} -3 & -2 \\ -\gamma & 1 \end{vmatrix} + \gamma \begin{vmatrix} 1 & -3 \\ 3 & -\gamma \end{vmatrix}$$

$$= 2(-3-2\gamma) + \gamma(-\gamma+9) = -\gamma^2 - 4\gamma + 9\gamma - 6$$

$\Rightarrow$ $\vec u, \vec v, \vec w$ are linearly dependent for

$$\gamma^2 - 5\gamma + 6 = 0 \qquad \Rightarrow \gamma = 2 \quad \text{or} \quad \gamma = 3$$

iii) (seen)

$$\vec u \times \vec v \times \vec w = \vec u \times \begin{vmatrix} \vec i & \vec j & \vec k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= \vec u \times \Big[ \underbrace{(v_2 w_3 - w_2 v_3)}_{a} \vec i + \underbrace{(v_3 w_1 - v_1 w_3)}_{b} \vec j + \underbrace{(v_1 w_2 - v_2 w_1)}_{c} \vec k$$

$$= \begin{vmatrix} \vec i & \vec j & \vec k \\ \gamma & 0 & 0 \\ a & b & c \end{vmatrix} = -\gamma c \vec j + \gamma b \vec k$$

$$= (-\gamma v_1 w_2 + \gamma v_2 w_1)\vec j + (\gamma v_3 w_1 - \gamma w_3)\vec k$$

$$= \gamma w_1 (v_2 \vec j + v_3 \vec k) - \gamma v_1 (w_2 \vec j + w_3 \vec k) + \underbrace{\gamma w_1 v_1 \vec i - \gamma w_1 v_1 \vec i}_{0}$$

$$= \underbrace{\gamma w_1}_{\vec u \cdot \vec w} \underbrace{(v_1 \vec i + v_2 \vec j + v_3 \vec k)}_{\vec v} - \underbrace{\gamma v_1}_{\vec u \cdot \vec v} \underbrace{(w_1 \vec i + w_2 \vec j + w_3 \vec k)}_{\vec w}$$

$$= (\vec u \cdot \vec w)\vec v - (\vec u \cdot \vec v)\vec w \qquad \square$$

(10)

iv) (seen)

$$(\vec u \times \vec v) \times \vec w \times \vec x = \Big[\underbrace{(\vec u \times \vec v)\cdot \vec x}_{\vec u \cdot (\vec v \times \vec x)}\Big]\vec w - \Big[\underbrace{(\vec u \times \vec v)\cdot \vec w}_{\vec u \cdot (\vec v \times \vec w)}\Big]\vec x$$

(3)

cyclic property of the scalar triple product

v) (seen) $(\vec u \times \vec v) \times (\vec w \times \vec x) = -(\vec w \times \vec x) \times (\vec u \times \vec v)$ antisymmetry of $\times$-product

$$\underset{iv)}{=} -[\vec w \cdot (\vec x \times \vec v)]\vec u + [\vec w \cdot (\vec x \times \vec u)]\vec v$$

$$= [\vec u \cdot (\vec v \times \vec x)]\vec w - [\vec u \cdot (\vec v \times \vec w)]\vec x$$

directly iv)

~~Solve this for $\vec{x}$:~~

$$\cancel{\vec{x} = \frac{\vec{w}\cdot(\vec{x}\times\vec{v})}{\vec{u}\cdot(\vec{v}\times\vec{w})}\vec{u} + \frac{\vec{w}\cdot(\vec{u}\times\vec{x})}{\vec{u}\cdot(\vec{v}\times\vec{w})}\vec{v} + \frac{\vec{u}\cdot(\vec{v}\times\vec{x})}{\vec{u}\cdot(\vec{v}\times\vec{w})}\vec{w}} \quad (4)$$

$\boxed{\Sigma = 25}$

8' i) The distance d of the centre $C_1$ of $S_1$, i.e. the origin to the plane P is:

$$d_1 = \left|\frac{-6}{\sqrt{\lambda^2+\mu^2+1}}\right|$$

For the point in P to be on the sphere as well we need

$$d_1 = 3 \Rightarrow 3\sqrt{\lambda^2+\mu^2+1} = 6$$

$$\Rightarrow \lambda^2+\mu^2+1 = 2^2 \Rightarrow \underline{\underline{\lambda^2+\mu^2 = 3}} \quad (8)$$

ii) - The centre of $S_2$ is $C_2(0,-6,0)$

- The distance $d_2$ of $C_2$ to P is

$$d_2 = \left|\frac{-6\mu-6}{\sqrt{\lambda^2+\mu^2+1}}\right|$$

- For this point to be on $S_2$ as we need $d_2 = 6$ (7)

$$\Rightarrow 6\sqrt{\lambda^2+\mu^2+1} = 6(\mu+1)$$

$$\Rightarrow \lambda^2+\mu^2+1 = \mu^2+2\mu+1 \Rightarrow \underline{\underline{\lambda^2 = 2\mu}}$$

iii) For P to be a tangent plane to $S_1$ and $S_2$ we have to solve

$$\lambda^2+\mu^2 = 3 \quad \wedge \quad \lambda^2 = 2\mu$$

$$\Rightarrow \mu^2+2\mu-3 = 0$$

$$\Rightarrow \mu_{1/2} = -1 \pm \sqrt{1+3} = -1 \pm 2$$

$\because \lambda^2 = 2\mu > 0$ discard the minus sign

$\Rightarrow \mu = 1 \quad \Rightarrow \lambda = \pm\sqrt{2}$

(8)

$\Rightarrow \mathcal{P}: \pm\sqrt{2}\,x + y + z = 6$ is the tangent plane to $S_1, S_2$

iv) For instance $A(0,0,6)$ and $B(0,6,0)$ are in $\mathcal{P}$.

$\Rightarrow \overrightarrow{AB} = (0, 6, -6) \in \mathcal{P}$

(2)

$\boxed{\Sigma = 25}$