Geometry & Vectors

Exercises 4

1) For the orthonormal basis $\vec{\imath}, \vec{\jmath}, \vec{k}$ choose a left handed orientation. Deduce the relations for all six possible vector products involving $\vec{\imath}, \vec{\jmath}, \vec{k}$ by using the only properties of the product.

2) The vectors $\vec{\imath}, \vec{\jmath}, \vec{k}$ constitute an orthonormal basis. For the vectors

$$\vec{u} = 2\vec{\imath} - \vec{\jmath} - 2\vec{k}, \qquad \vec{v} = \vec{\imath} - 2\vec{\jmath} + 3\vec{k}, \qquad \text{and} \qquad \vec{w} = -2\vec{\imath} + \vec{\jmath} - 5\vec{k}$$

i) compute the expressions

$$(\vec{u} \cdot \vec{v})\, \vec{w}, \qquad (\vec{u} \cdot \vec{w})\, \vec{v}, \qquad \vec{u} \cdot \vec{v} \times \vec{w}, \qquad \vec{w} \cdot \vec{u} \times \vec{v} \qquad \text{and} \qquad \vec{u} \times \vec{v} \times \vec{w}\ .$$

ii) Use your results from i) to verify that

$$\vec{u} \times \vec{v} \times \vec{w} = (\vec{u} \cdot \vec{w})\, \vec{v} - (\vec{u} \cdot \vec{v})\, \vec{w}\ .$$

3) Show that the cross product satisfies the Jacobi identity

$$\vec{u} \times \vec{v} \times \vec{w} + \vec{w} \times \vec{u} \times \vec{v} + \vec{v} \times \vec{w} \times \vec{u} = 0.$$

4) The vectors $\vec{\imath}, \vec{\jmath}, \vec{k}$ constitute an orthonormal basis. Determine the angle between the two vectors

$$\vec{u} = 5\vec{\imath} - \vec{\jmath} - 2\vec{k} \qquad \text{and} \qquad \vec{v} = 3\sqrt{5}\vec{\imath} - 2\vec{\jmath} + \vec{k}$$

in two alternative ways using

$$|\vec{u} \times \vec{v}| = |\vec{u}|\, |\vec{v}| \sin\theta \qquad \text{and} \qquad \vec{u} \cdot \vec{v} = |\vec{u}|\, |\vec{v}| \cos\theta\ .$$

Verify that the answers are identical.

5) The vectors $\vec{\imath}, \vec{\jmath}, \vec{k}$ constitute an orthonormal basis. For the vectors

$$\vec{u} = \vec{\imath} - 3\vec{\jmath} - 2\vec{k}, \qquad \vec{v} = 3\vec{\imath} - \eta\vec{\jmath} + \vec{k}, \qquad \text{and} \qquad \vec{w} = 2\vec{\imath} + \eta\vec{k}$$

compute $\vec{u} \cdot (\vec{v} \times \vec{w})$ and decide for which values of η the vectors $\vec{u}, \vec{v}, \vec{w}$ become linearly dependent.

6) Consider three points U, V, W with position vectors $\vec{u}, \vec{v}, \vec{w}$. Prove that the three points are collinear if and only if

$$\vec{u} \times \vec{v} + \vec{v} \times \vec{w} + \vec{w} \times \vec{u} = 0$$

holds.

Solutons exercises 4

1) An argumentation similar to the one in the lecture for the right handed orientation gives

$$\vec{\imath}\times\vec{\jmath} = -\vec{k}, \qquad \vec{k}\times\vec{\imath} = -\vec{\jmath}, \qquad \vec{\jmath}\times\vec{k} = -\vec{\imath}, \qquad \vec{\jmath}\times\vec{\imath} = \vec{k}, \qquad \vec{\imath}\times\vec{k} = \vec{\jmath}, \qquad \vec{k}\times\vec{\jmath} = \vec{\imath}.$$

2) i)

$$\begin{aligned}
(\vec{u}\cdot\vec{v})\,\vec{w} &= 4\vec{\imath} - 2\vec{\jmath} + 10\vec{k}, \\
(\vec{u}\cdot\vec{w})\,\vec{v} &= 5\vec{\imath} - 10\vec{\jmath} + 15\vec{k}, \\
\vec{u}\cdot\vec{v}\times\vec{w} &= \vec{w}\cdot\vec{u}\times\vec{v} = 21 \\
\vec{u}\times\vec{v}\times\vec{w} &= \vec{\imath} - 8\vec{\jmath} + 5\vec{k}.
\end{aligned}$$

ii) $\Rightarrow$

$$\vec{u}\times\vec{v}\times\vec{w} = (\vec{u}\cdot\vec{w})\,\vec{v} - (\vec{u}\cdot\vec{v})\,\vec{w}\ .$$

3) Add

$$\begin{aligned}
\vec{u}\times\vec{v}\times\vec{w} &= (\vec{u}\cdot\vec{w})\,\vec{v} - (\vec{u}\cdot\vec{v})\,\vec{w} \\
\vec{w}\times\vec{u}\times\vec{v} &= (\vec{w}\cdot\vec{v})\,\vec{u} - (\vec{w}\cdot\vec{u})\,\vec{v} \\
\vec{v}\times\vec{w}\times\vec{u} &= (\vec{v}\cdot\vec{u})\,\vec{w} - (\vec{v}\cdot\vec{w})\,\vec{u}\ .
\end{aligned}$$

4)

$$|\vec{u}\times\vec{v}| = |\vec{u}|\,|\vec{v}|\sin\theta \qquad \text{and} \qquad \vec{u}\cdot\vec{v} = |\vec{u}|\,|\vec{v}|\cos\theta\ .$$

$$\left.\begin{array}{l} |\vec{u}| = \sqrt{30} \\ |\vec{v}| = 5\sqrt{2} \\ \vec{u}\cdot\vec{v} = 375 \end{array}\right\} \Rightarrow \theta = \arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

$$\left.\begin{array}{l} |\vec{u}| = \sqrt{30} \\ |\vec{v}| = 5\sqrt{2} \\ |\vec{u}\times\vec{v}| = 5\sqrt{15} \end{array}\right\} \Rightarrow \theta = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

5) $\vec{u}\cdot(\vec{v}\times\vec{w}) = -\eta^2 + 5\eta - 6 = 0 \Rightarrow \eta = 2$ or 3.

6) The three points are on one line, if and only if the angle between the two vectors $(\vec{v}-\vec{u})$ and $(\vec{w}-\vec{u})$ is zero. Then

$$(\vec{v}-\vec{u})\times(\vec{w}-\vec{u}) = 0 \qquad \Leftrightarrow \qquad \vec{u}\times\vec{v} + \vec{v}\times\vec{w} + \vec{w}\times\vec{u} = 0.$$