**604.03**

# CITY UNIVERSITY

## London

BSc Degrees in Mathematical Science
Mathematical Science with Statistics
Mathematical Science with Computer Science
Mathematical Science with Finance and Economics
MMath Degrees in Mathematical Science

PART III EXAMINATION

# Geometry and Vectors

May 2005 9:00 am – 11:00 am

Time allowed: 2 hours

*Full marks may be obtained for correct answers to THREE of the FIVE questions.*

*If more than THREE questions are answered, the best THREE marks will be credited.*

Turn over . . .

## Section **A**

Answer **all six** questions from this section. Each question carries 8 marks.

**1.** ABC constitutes a triangle. The point $P$ is situated on the line segment $\overline{BC}$, the point $Q$ is on the line segment $\overline{AC}$ and the point $R$ is on the line segment $\overline{AB}$, such that

$$\overline{BP} = \frac{3}{4}\,\overline{BC}, \qquad \overline{CQ} = \frac{3}{4}\,\overline{CA} \qquad \text{and} \qquad \overline{AR} = \frac{3}{4}\,\overline{AB}.$$

Sketch the corresponding figure and show that the sum of the vectors pointing from the corners $A, B$ and $C$ to the points $P, Q, R$, respectively, is zero

$$\overrightarrow{AP} + \overrightarrow{BQ} + \overrightarrow{CR} = 0.$$

**2.** The vectors $\vec{\imath}, \vec{\jmath}, \vec{k}$ constitute an orthonormal basis. Given are the vectors

$$\vec{u} = 2\vec{\imath} - \vec{\jmath} + \vec{k}, \qquad \text{and} \qquad \vec{v} = 3\vec{\imath} - 3\vec{\jmath}$$

$(i)$ Determine the angle between $\vec{u}$ and $\vec{v}$.

$(ii)$ Construct all unit vectors which are perpendicular to $\vec{u}$ and $\vec{v}$.

$(iii)$ Verify the triangle inequality for the vectors $\vec{u}$ and $\vec{v}$.

**3.** The vectors $\vec{\imath}, \vec{\jmath}, \vec{k}$ constitute an orthonormal basis. For the vectors

$$\vec{u} = \vec{\jmath} + 2\vec{k}, \quad \vec{v} = -\vec{\imath} + 3\vec{\jmath}, \quad \vec{w} = \vec{\imath} - \vec{\jmath} \quad \text{and} \quad \vec{x} = \vec{\imath} + \vec{\jmath} - \vec{k}$$

verify the identity

$$(\vec{u} \times \vec{v}) \cdot (\vec{w} \times \vec{x}) = (\vec{u} \cdot \vec{w})(\vec{v} \cdot \vec{x}) - (\vec{u} \cdot \vec{x})(\vec{v} \cdot \vec{w}) \ .$$

**4.** An ellipse is parameterized by the equation

$$2x^2 + 3y^2 - 4x + 5y + 4 = 0 \ .$$

In the $x, y$-plane find the centre of the ellipse, the length of the major and minor axis, the location of the foci and vertices, its eccentricity and the equation of the directrix.

Turn over ...

**5.** Given are the three points A(7,1,2), B(9,-2,7) and C(11,-1,5).

(*i*) Determine the equation of the plane which contains the three points A,B and C.

(*ii*) Show that the shortest distance of the point P(16,15,9) to this plane is $3\sqrt{29}$.

**6.** A line passes through the points P(-5,4,1) and Q(1,3,2). Determine the point of intersection of this line with the plane described by

$$3x - y + 6z - 87 = 0 \ .$$

Turn over . . .

Section **B**

Answer **two** questions from this section. Each question carries 26 marks.

**7.** (*i*) State Euclid's five axioms.

(*ii*) Use the axioms to prove the following:

Given a line $\mathcal{L}$ and a point $P$ which is not on the line, i.e. $P \notin \mathcal{L}$, there is a unique plane $\mathcal{P}$ containing both $\mathcal{L}$ and $P$.

**8.** $U, V, W$ and $X$ are four arbitrary points in $\mathbb{R}^3$ with position vectors $\vec{u}, \vec{v}, \vec{w}$ and $\vec{x}$, respectively. Prove the following identities:

(*i*)

$$\overrightarrow{XU} \cdot \overrightarrow{VW} + \overrightarrow{XV} \cdot \overrightarrow{WU} + \overrightarrow{XW} \cdot \overrightarrow{UV} = 0$$

(*ii*)

$$\left|\overrightarrow{UV}\right|^2 - \left|\overrightarrow{VW}\right|^2 + \left|\overrightarrow{WX}\right|^2 - \left|\overrightarrow{XU}\right|^2 = 2\overrightarrow{UW} \cdot \overrightarrow{XV}$$

(*iii*)

$$\vec{u} \times \vec{v} \times \vec{w} = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$$

(Hint: Use an orthonomal basis such that $\vec{u} = \gamma\vec{\imath}$ with $\gamma \in \mathbb{R}$.)

(*iv*)

$$\vec{u} \times \vec{v} \times \vec{w} \times \vec{x} = (\vec{u} \cdot (\vec{v} \times \vec{x}))\vec{w} - (\vec{u} \cdot (\vec{v} \times \vec{w}))\vec{x}$$

(Hint: Use the identity from (*iii*).)

**9.** Given are the five lines

$$\begin{aligned}
\mathcal{L}_1 &: \quad y = \frac{1}{\sqrt{3}}x \\
\mathcal{L}_2 &: \quad y = -\frac{1}{\sqrt{3}}x \\
\mathcal{L}_3 &: \quad \frac{x-5}{3} = \frac{y-1}{2} = z+1 \\
\mathcal{L}_4 &: \quad \frac{x-2}{3} = y+1 = \frac{1-z}{4} \\
\mathcal{L}_5 &: \quad \frac{x+1}{2} = \frac{y-2}{3} = z+2\ .
\end{aligned}$$

(*i*) Determine the angle between the lines $\mathcal{L}_1$ and $\mathcal{L}_2$.

Turn over . . .

(*ii*) Transform the hyperbola

$$f(r,\theta) = r + \frac{\sqrt{3}}{1 - e\cos\theta} = 0$$

to the nomal form such that the lines $\mathcal{L}_1$ and $\mathcal{L}_2$ are the asymptotes.

(*iii*) Find the equation of the plane which contains the line $\mathcal{L}_3$ and which is parallel to the line $\mathcal{L}_4$.

(*iv*) Find the point in which the lines $\mathcal{L}_3$ and $\mathcal{L}_5$ intersect.

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