MA1607

CITY UNIVERSITY

London

BSc Degrees in Mathematical Science
Mathematical Science with Statistics
Mathematical Science with Computer Science
Mathematical Science with Finance and Economics
MMath Degrees in Mathematical Science

PART I EXAMINATION

Geometry and Vectors

25th of May 2007

Time allowed: 2 hours

Full marks may be obtained for correct answers to FIVE questions of section A and TWO questions of section B.

If more than TWO questions are answered of section B, the best TWO marks will be credited.

Turn over . . .

Section **A**

Answer **all five** questions from this section. Each question carries 10 marks.

1. The vectors $\vec{\imath}, \vec{\jmath}, \vec{k}$ constitute an orthonormal basis in Euclidean space. A and B are points with position vectors $\overrightarrow{OA} = \frac{3}{2}\vec{\imath} + \vec{\jmath}$ and $\overrightarrow{OB} = 3\vec{\imath} + 4\vec{\jmath}$.

(i) Find the position vector for a point C on the line through A and B, i.e. $C \in \overleftrightarrow{AB}$ such that $AC : CB = 2 : 1$.

(ii) What does it mean to say the two points X and Y on a line $\mathcal{L}$ divide AB harmonically?

(iii) Find the position vector for a point D, such that C and D divide AB harmonically.

2. Given are the three vectors

$$\begin{aligned}\vec{u} &= \frac{1}{2}\left(\tau_+\vec{\imath} - \vec{\jmath} + \tau_-\vec{k}\right),\\ \vec{v} &= \frac{1}{2}\left(-\tau_+\vec{\imath} + \vec{\jmath} + \tau_-\vec{k}\right),\\ \vec{w} &= \frac{1}{2}\left(\vec{\imath} + \tau_-\vec{\jmath} - \tau_+\vec{k}\right),\end{aligned}$$

where $\tau_+ = \frac{1}{2}(\sqrt{5}+1)$ and $\tau_- = \frac{1}{2}(\sqrt{5}-1)$.

(i) Show that $\vec{u}$, $\vec{v}$ and $\vec{w}$ are unit vectors.

(ii) Determine all three angles inside the triangle formed by the two vectors $\vec{u}$ and $\vec{v}$.

(iii) Find the angle between $\vec{w}$ and $\vec{u}$ and also the one between $\vec{w}$ and $\vec{v}$.

(Hint: The computations simplify if you make use of the fact that τ_+ is the golden ratio and τ_- its inverse, such that the following identities can be used $\tau_+^2 = 1 + \tau_+$, $\tau_-^2 = 1 - \tau_-$ and $\tau_-\tau_+ = 1$. You can also use $\cos(4\pi/5) = -\,\tau_+/2$.)

Turn over ...

3. An electron with charge q is sent with velocity $\vec{v}$ into a magnetic field, where it is subject to the Lorentz force $\vec{F} = q(\vec{v} \times \vec{B})$. The vector $\vec{B}$ is the magnetic induction.

(i) In two experiments, in which the electron is send into different directions, the force $\vec{F}$ is measured. In experiment 1 the electron has velocity $\vec{v} = 2\vec{\imath} + \vec{\jmath}$ and the force $\vec{F} = q(-2\vec{\imath} + 4\vec{\jmath} - 5\vec{k})$ is measured. In experiment 2 the electron has velocity $\vec{v} = 3\vec{\imath} - 2\vec{\jmath} + 5\vec{k}$ and the force $\vec{F} = q(19\vec{\imath} + \vec{\jmath} - 11\vec{k})$ is measured. Determine the magnetic induction $\vec{B}$ from these data .

(ii) In a third experiment one measures for the same magnetic induction $\vec{B}$ the force $\vec{F} = q(6\vec{\imath} + 2\vec{\jmath} - 6\vec{k})$, but forgets to record the velocity. Could the electron have had the velocity $\vec{v} = \vec{\imath} - 3\vec{\jmath}$ or $\vec{v} = 2\vec{\imath} + 2\vec{k}$? Could the electron have come from the direction $\vec{\imath} + \vec{k}$?

(iii) Is it possible to design an experiment such that only one measurement is required to find $\vec{B}$?

4. Find the gradients of the tangents to the ellipse with equation

$$5x^2 + 9y^2 = 180,$$

passing through the point $P(9, 0)$.

5. Given are the two points $A(0, 4, -3)$ and $B(1, 2, 1)$

(i) Find the equation of the line passing through A and B. Subsequently determine the point of intersection of this line with the xz-plane.

(ii) Find the coordinates of the point in which the line through the points A and B intersects the plane

$$\mathcal{P}: \quad 5x - 3y + 2z = 1.$$

Turn over ...

Section **B**

Answer **two** questions from this section. Each question carries 25 marks.

6. (*i*) Prove that the distance d of a point $P(x_0, y_0)$ from a line $\mathcal{L}$ described by the equation $\alpha x + \beta y + \gamma = 0$ is given by

$$d = \left| \frac{\alpha x_0 + \beta y_0 + \gamma}{\sqrt{\alpha^2 + \beta^2}} \right| ,$$

where $\alpha, \beta, \gamma \in \mathbb{R}$.

(*ii*) Given are the two lines

$$\mathcal{L}_1 \ : \ 2y - 5x - 6 = 0$$
$$\mathcal{L}_2 \ : \ 4y - 10x + 8 = 0.$$

Determine the distance between these two lines. State Euclid's axiom which allows you to use the formula proven in (*i*).

7. (*i*) Why is $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$ the same as saying the vectors $\vec{u}, \vec{v}$ and $\vec{w}$ are linearly dependent?

(*ii*) The vectors $\vec{\imath}, \vec{\jmath}, \vec{k}$ constitute an orthonormal basis. For the vectors

$$\vec{u} = \vec{\imath} - 3\vec{\jmath} - 2\vec{k}, \qquad \vec{v} = 3\vec{\imath} - \gamma\vec{\jmath} + \vec{k}, \qquad \text{and} \qquad \vec{w} = 2\vec{\imath} + \gamma\vec{k}.$$

decide for which values of γ the vectors $\vec{u}, \vec{v}, \vec{w}$ become linearly dependent.

(*iii*) Prove the identity

$$\vec{u} \times \vec{v} \times \vec{w} = (\vec{u} \cdot \vec{w})\, \vec{v} - (\vec{u} \cdot \vec{v})\, \vec{w}$$

Hint: Without loss of generality you may choose your coordinate system such that $\vec{u} = \gamma\vec{\imath}$ with $\gamma \in \mathbb{R}$.

(*iv*) Prove the identity

$$\vec{u} \times \vec{v} \times \vec{w} \times \vec{x} = [\vec{u} \cdot (\vec{v} \times \vec{x})]\, \vec{w} - [\vec{u} \cdot (\vec{v} \times \vec{w})]\, \vec{x}.$$

Hint: Use for this the identity from (*iii*).

Turn over ...

8. Given are the two spheres

$$\mathcal{S}_1 \;:\; x^2 + y^2 + z^2 = 9$$
$$\mathcal{S}_2 \;:\; x^2 + (y+6)^2 + z^2 = 36$$

and the plane

$$\mathcal{P}: \; \lambda x + \mu y + z = 6.$$

(*i*) Show that the condition for $\mathcal{P}$ to be the tangent plane to $\mathcal{S}_1$ is

$$\lambda^2 + \mu^2 = 3$$

(*ii*) Find the condition for $\mathcal{P}$ to be also the tangent plane to $\mathcal{S}_2$.

(*iii*) Determine the equation of the plane which is the common tangent plane to $\mathcal{S}_1$ and $\mathcal{S}_2$.

(*iv*) Specify a vector which lies in the plane determined in (*iii*).

Internal Examiner: Dr. A. Fring
External Examiners: Professor M.A. O'Neill
Professor J. Billingham