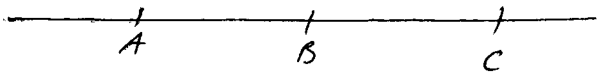
where we did not yet say what we mean by midpoint:

<u>Def.</u>: The *midpoint* of the line segment AC is defined to be the unique point B on AC such that

$AB : BC = 1 : 1$



2) Vectors

2.1) The notion of a vector

There are scalar quantities in physics such as masses, length, volumes, temperatures, energies, ..., i.e. they are just numbers with some units attached (kg, m, C°, J ...). There are other quantities which besides being associated to numbers also have a direction attached to them, such as velocities, accelerations, forces, electric and magnetic fields, electric currents, etc.

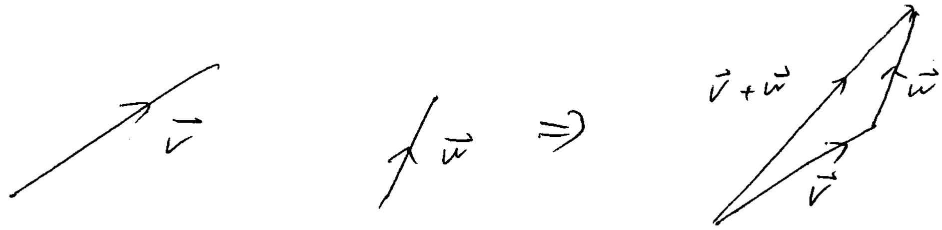
Thus one may think of a vector as having a <u>magnitude</u> as well as a <u>direction</u>.

One can depict a vector as an arrow on a line segment

$\overrightarrow{AB}$ or $\underline{AB}$ or $\vec{AB} = \vec{v}$

Note that $\overrightarrow{AB} \neq \overleftarrow{AB}$!

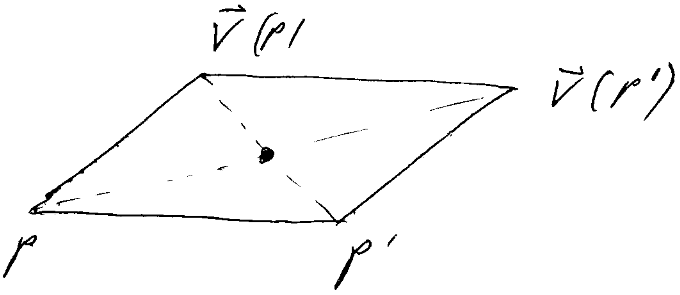
Adding two vectors then corresponds to "shifting" arrows.



This is not very precise and one can make this mathematically more rigorous:

One can think of vectors as instructions to move into a certain direction, which is mathematically a map.

<u>Def.</u>: A vector is a mapping $\vec V$ $(\mathbb{R}^n \to \mathbb{R}^n)$ which associates to each point P a new point $\vec V(P)$ having the property that for any two points P and P' the midpoint of $P\,\vec V(P')$ is equal to the midpoint of $P'\,\vec V(P)$



Thus by our previous definition $P\;\vec V(P')\;\vec V(P)\;P'$ is a parallelogram.

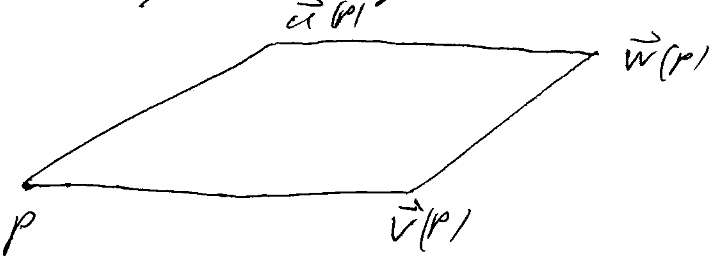
<u>Def.</u>: The <u>nullvector</u> $\vec O$ has the property $\vec O(P) = P$ for any point P.

Prop.: For any two points P and P' there is exactly one vector such that $\vec{v}(P) = P'$. (uniqueness!)

Proof: Omitted

2.2) Addition and Scaling of vectors

Prop.: The composition of two vectors $\vec{v}$ and $\vec{u}$ defines a new mapping $\vec{w}(P) = \vec{v}(\vec{u}(P))$ which is again a vector, such that P $\vec{u}(P)$ $\vec{w}(P)$ $\vec{v}(P)$ constitutes a parallelogram.



Proof: Omitted

Def.: The addition of two vectors is defined as their composition:

$$\vec{w}(P) = \vec{u}(\vec{v}(P)) =: \vec{u} + \vec{v}$$

Notice this is more precise than the previous recipe.

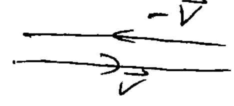
Properties of vector addition:

i) For any given vector $\vec{v}$ we have

$$\boxed{\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v}} \qquad (A1)$$

ii) For any given vector $\vec{v}$ there exists a unique vector $-\vec{v}$ such that

$$\boxed{\vec{v} + (-\vec{v}) = \vec{0}} \qquad (A2)$$

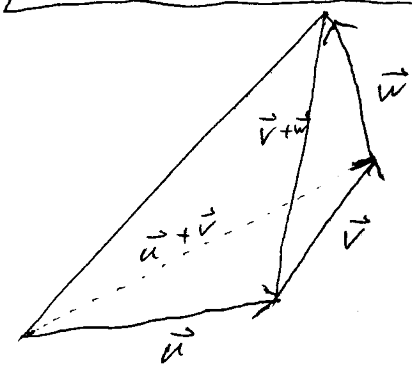


iii) For any given vectors $\vec{u}$ and $\vec{v}$ we have

$$\boxed{\vec{v} + \vec{u} = \vec{u} + \vec{v}} \qquad (A3)$$

iv) For any given vectors $\vec{u}, \vec{v}, \vec{w}$ we have associativity

$$\boxed{(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})} \qquad (A4)$$

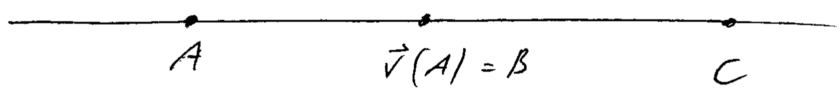


- We omit the proofs for these rules. (A4 we prove in section 2.6)

We can also multiply vectors by scalars (real numbers) and scale their magnitude by leaving the direction unchanged.

Def.: The map $\lambda \vec{v}$ for $\lambda \in \mathbb{R}$ is understood as follows: If $\vec{v}(A) = B$, then $(\lambda \vec{v}) = C$, where C is the unique point on $\overleftrightarrow{AB}$ for which

$$AB : AC = 1 : \lambda$$



In other words $\lambda \vec{v}$ is the vector which sends the point A into the same direction as $\vec{v}(A)$ but λ times as far.

Scaling properties of vectors

For any two vectors $\vec{u}$ and $\vec{v}$ and scalars λ and μ we have:

i) Associative law: $\boxed{(\lambda\mu)\vec{v} = \lambda(\mu\vec{v})}$ (S1)

ii) Distributive laws:

$$\boxed{\begin{aligned} (\lambda+\mu)\vec{V} &= \lambda\vec{v} + \mu\vec{V} \\ \lambda(\vec{u}+\vec{v}) &= \lambda\vec{u} + \lambda\vec{v} \end{aligned}} \qquad (S2)$$

iii) Particular scalar multiples

$$\boxed{\begin{aligned} 0\,\vec{V} &= \vec{0} \\ 1\,\vec{V} &= \vec{V} \\ (-1)\,\vec{v} &= -\vec{V} \end{aligned}} \qquad (S3)$$

2.3) Vectorspaces

Def.: The space made up from the collection of vectors equipped with the binary operations addition and scalar multiplication which obey the rules (A1)-(A4) and (S1)-(S3) is called a *vectorspace* V.

Def.: A set of vectors $\vec{V}_1, \dots \vec{V}_n \in V$ is *linearly dependent* if there are scalars $\lambda_1, \dots, \lambda_n$ not equal to zero, such that

$$\sum_{i=1}^{n} \vec{v}_i = \vec{0} .$$

Otherwise they are *linearly independent*.

Def.: A linearly independent set of vectors $\vec{e}_1, \dots, \vec{e}_n$ is called a *basis* of V, if any vector $\vec{v} \in V$ can be expressed as

$$\vec{V} = \lambda_1 \vec{e}_1 + \dots + \lambda_n \vec{e}_n$$

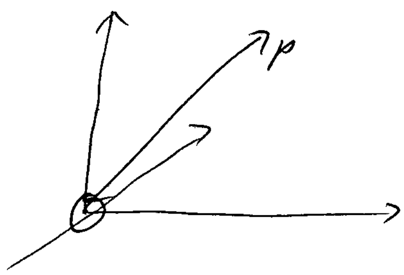
The coefficients λ_i for $1 \leq i \leq n$ are unique and are called the <u>components</u> of $\vec{v}$.

Def.: The number of basis vectors needed to span V is called the <u>dimension</u> of V. — (see algebra course for more justification)

2.4) The <u>length</u> of <u>a vector</u>

So far all vectors can be moved around freely. Now we choose a particular point O in space and call it the <u>origin</u>.

Def.: The vector $\vec{OP}$ is the <u>position vector</u> of the point P in relation to the origin



The choice of O is arbitrary.

First we need to know what is the length of a line.

We select for this a standard unit of our scale, such as m, inches, .. which defines uniquely two points on a line with components 0 and 1. Then the <u>distance</u> between two points A, B with coordinates a, b (or the <u>length</u> of the line segment AB) is $|a-b|$.

Axiom 10 (ruler compatibility axiom)

For any line L there is at least one ruler on L such that for all points $A, B \in L$

$$|AB| = |a-b|$$

where a and b are the coordinates relative to the given ruler.

Such type of ruler is called a standard ruler.

Length of a line $\Rightarrow$ length of a vector?

Def.: The length $|\vec{v}|$ of a vector $\vec{v}$ is the distance $|A\,\vec{v}(A)|$ for any point A. A vector of length 1 is called a unit vector.

In other words: The length is the distance through which $\vec{v}$ moves any general point A.

Do we always get the same result, even if we start at different points A? (This can be guaranteed by the congruency axiom, which we omit here.)

Properties:

$$|\vec{XY}| = |XY|$$

$$|\lambda \vec{v}| = |\lambda||\vec{v}| \qquad \lambda \in \mathbb{R}$$

$$|-\vec{v}| = |\vec{v}|$$

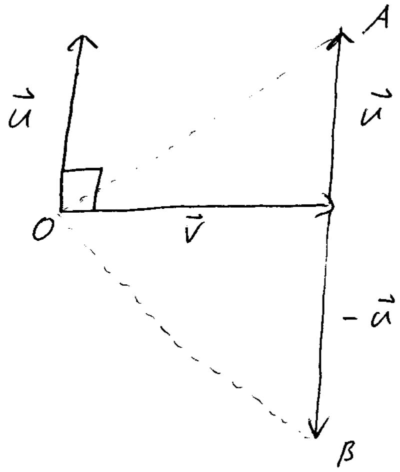
position vector $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

($\vec{i}, \vec{j}, \vec{k}$ are an orthogonal basis; $\vec{i}\cdot\vec{i}=1$, $\vec{i}\cdot\vec{j}=0, \ldots$)

$$|\vec{r}| = \sqrt{x^2+y^2+z^2} \quad \text{(proof later)}$$

Def.: Two vectors $\vec{u}$ and $\vec{v}$ are said to be perpendicular if (orthogonal)

$$|\vec{u} + \vec{v}| = |\vec{u} - \vec{v}|$$



$$|OA| = |OB|$$

Expl.: $P = (5,0)$ $\quad Q = (0,9)$

$\vec{OP} = 5i \qquad \vec{OQ} = 9j$

$|\vec{OP} + \vec{OQ}| = \sqrt{5^2 + 9^2} = \sqrt{106}$

$|\vec{OP} - \vec{OQ}| = \sqrt{5^2 + (-9)^2} = \sqrt{106}$

2.5) The Dot or Scalar Product (angles)

The dot (or scalar product) defines the product between two vectors $\vec{u}$ and $\vec{v}$. We write $\quad \vec{u} \cdot \vec{v}$

The product has the following properties ($\lambda, \mu \in \mathbb{R}$)

$$\boxed{\begin{array}{ll} (\lambda \vec{u}_1 + \mu \vec{u}_2) \cdot \vec{v} = \lambda \vec{u}_1 \cdot \vec{v} + \mu \vec{u}_2 \cdot \vec{v} & \text{linear} \\ \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} & \text{symmetric} \\ \vec{u} \cdot \vec{u} \geq 0 & \text{positive} \\ \vec{u} \cdot \vec{0} = \vec{0} \cdot \vec{u} = 0 & \text{definite} \end{array}} \quad (P)$$

In components: ($\vec{i}, \vec{j}, \vec{k}$ form an orthonormal basis)

$$\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$$

$$\vec{u} = u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k}$$

$$\vec{v} \cdot \vec{u} = (v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}) \cdot (u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k})$$

$$= (v_1 u_1)\underbrace{(\vec{i}\cdot\vec{i})}_{1} + (v_2 u_2)\underbrace{(\vec{j}\cdot\vec{j})}_{1} + v_3 u_3 \underbrace{(\vec{k}\cdot\vec{k})}_{1} + v_1 u_2 \underbrace{\vec{i}\cdot\vec{j}}_{0} + \dots 0$$

$$\boxed{\vec{v} \cdot \vec{u} = v_1 u_1 + v_2 u_2 + v_3 u_3 = \sum_{i=1}^{3} v_i u_i}$$

The length of a vector is then also

$$\boxed{|\vec{v}| = \sqrt{\vec{v}\cdot\vec{v}}}$$

Example: (polarisation identity)

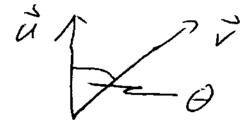
$$\begin{aligned}
|\vec{u} - \vec{v}|^2 &= (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) \\
&= \vec{u} \cdot (\vec{u} - \vec{v}) - \vec{v} \cdot (\vec{u} - \vec{v}) \\
&= \vec{u}\cdot\vec{u} - \vec{u}\,\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\;\vec{v} \\
&= |\vec{u}|^2 + |\vec{v}|^2 - 2\,\vec{u}\cdot\vec{v}
\end{aligned}$$

$\Rightarrow$ polarisation identity: $\boxed{\vec{u}\cdot\vec{v} = \frac{1}{2}\left(|\vec{u}|^2 + |\vec{v}|^2 - |\vec{u}-\vec{v}|^2\right)}$

We can now use the dot product to define angles:

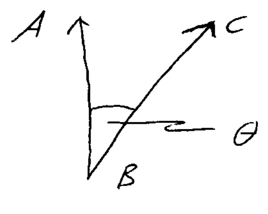
<u>Def.</u>: The <u>angle</u> between two non-zero vectors $\vec{u}$ and $\vec{v}$ is the unique value θ between 0 and π such that

$$\boxed{\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|}}$$



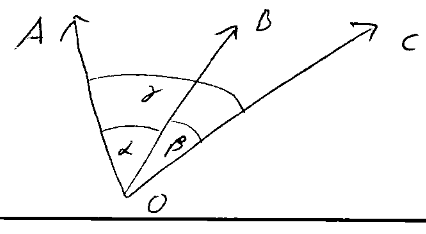
(assume that we know trigonometry)

- For three points A, B, C we denote $\theta = \widehat{ABC}$ to be the angle between $\vec{BA}$ and $\vec{BC}$



- We can add angles

$$\widehat{AOC} = \widehat{AOB} + \widehat{BOC}$$



or $\alpha + \beta = \gamma$

$\Longrightarrow$ <u>Expl.</u>:

Expl.: Find the angle between the two vectors

$$\vec{v} = 3\vec{i} + 2\vec{j} - 6\vec{k} \quad \text{and} \quad \vec{u} = 2\vec{i} - 3\vec{j} + \vec{k}$$

Compute $|\vec{v}| = \sqrt{3^2+2^2+6^2} = \sqrt{9+4+36} = 7$

$$|\vec{u}| = \sqrt{2^2+3^2+1} = \sqrt{14} =$$

$$\vec{v}\cdot\vec{u} = 6 - 6 - 6 = -6$$

$$\Rightarrow \cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|} = \frac{-6}{7\cdot\sqrt{14}} \quad \Rightarrow \arccos\left(\frac{-6}{7\cdot\sqrt{14}}\right) = \begin{cases} 0.5736\pi \\ 103.24^\circ \end{cases} = \theta$$

⇒ Alternative notion of perpendicular: $u \perp \vec{v}$ if the angle between $\vec{u}$ and $\vec{v}$ is $\frac{\pi}{2}$ that is $\vec{u} \cdot \vec{v} = 0$. (from definition)

– Compare with previous definition (section 2.4)

from polarisation identity compute

$$|\vec{u} - \vec{v}|^2 - |\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}| - 2\vec{u} \cdot \vec{v} + (-|\vec{u}|^2 - |\vec{v}| - 2\vec{u} \cdot \vec{v})$$

$$= -4\,\vec{u} \cdot \vec{v}$$

$$\Rightarrow \quad |\vec{u} - \vec{v}| = |\vec{u} + \vec{v}| \quad \Leftrightarrow \quad \vec{u} \cdot \vec{v} = 0$$

⇒ the two definitions are compatible

→ Theorem: → 16'

2.6) Euclidean geometry

Def.: A space $\mathcal{E}$ is called a Euclidean space if

i) for any two points $A, B \in \mathcal{E}$ there is a unique vector $\vec{AB}$

ii) for any two vectors we have

$$(\vec{u} + \vec{v})(X) = \vec{v}(\vec{u}(X)) \qquad X \in \mathcal{E}$$

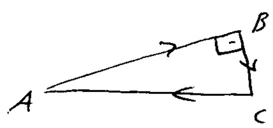
iii) there is a vector space associated to $\mathcal{E}$ with a dot product obeying (P).

Let's do some geometry!

Theorem: (Pythagoras)

For a triangle ABC with $\widehat{ABC} = \frac{\pi}{2}$ we have

$$|AB|^2 + |BC|^2 = |AC|^2$$



Proof:

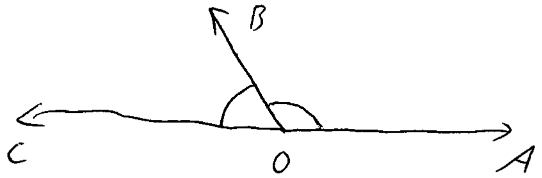
$$\vec{AC} = \vec{AB} + \vec{BC}$$

$$\Rightarrow |\vec{AC}|^2 = \vec{AC} \cdot \vec{AC} = (\vec{AB} + \vec{BC}) \cdot (\vec{AB} + \vec{BC})$$

$$= |\vec{AB}|^2 + |\vec{BC}|^2 + 2\underbrace{\vec{AB} \cdot \vec{BC}}_{0} \quad \text{if } \vec{AB} \perp \vec{BC}$$

Def.: Two angles are said to be supplementary if they add up to π

Expl.: Take $\vec{OA} = -\gamma\, \vec{OC}$ with $\gamma \in \mathbb{R}^+$



$$\vec{OA} \cdot \vec{OB} = \cos \widehat{AOB}\ |\vec{OA}|\ |\vec{OB}| = -\gamma\, \vec{OC} \cdot \vec{OB}$$

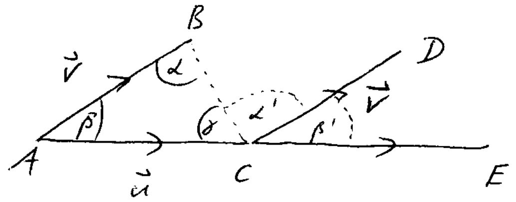
$$\vec{OC} \cdot \vec{OB} = \cos \widehat{BOC}\ |\vec{OC}|\ |\vec{OB}| =$$

$$\Rightarrow \cos \widehat{AOB} = -\cos \widehat{BOC}$$

$$\Rightarrow \widehat{AOB} + \widehat{BOC} = \pi$$

Theorem: The angles inside a triangle add up to π.

Proof:



$\vec{v} = \vec{AB} = \vec{CD}$

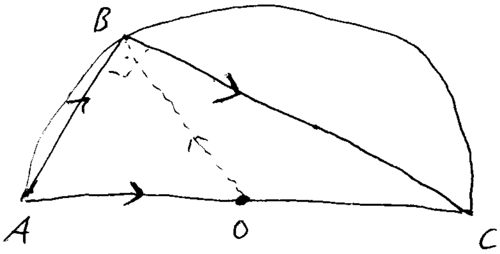
$\vec{u} = \vec{AC} = \vec{CE}$

$$\left.\begin{aligned} \cos \alpha &= \frac{-\vec{v} \cdot (\vec{u} - \vec{v})}{|\vec{v}|\ |\vec{u} - \vec{v}|} \\ \cos \alpha' &= -\frac{(\vec{u} - \vec{v}) \cdot \vec{v}}{|\vec{v}|\ |\vec{u} - \vec{v}|} \end{aligned}\right\} \Rightarrow \underline{\alpha = \alpha'}$$

$$\left.\begin{aligned} \cos \beta &= \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\ |\vec{v}|} \\ \cos \beta' &= \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|} \end{aligned}\right\} \Rightarrow \beta = \beta'$$

$(\alpha' + \beta')$ and γ are supplementary $\Rightarrow \underline{\alpha + \beta + \gamma = \pi}$

Theorem: Let A, B, C to be three points in a plane and O to be the midpoint of the line segment AC. If $|OA| = |OB| = |OC|$ then $\widehat{ABC} = \pi/2$.



Proof:

$$\vec{AB} = \vec{AO} + \vec{OB} \qquad \vec{BC} = \vec{BO} + \vec{OC}$$

$$\vec{AB}\cdot\vec{BC} = \underline{\vec{AO}\cdot\vec{BO}} + \underbrace{\vec{AO}\cdot\vec{OC}}_{|\vec{AO}|^2} + \underbrace{\vec{OB}\cdot\vec{BO}}_{-|\vec{OB}|^2} + \underbrace{\vec{OB}\cdot\vec{OC}}_{-\vec{BO}\cdot\vec{AO}} = 0$$

■

Proposition: (Cauchy-Schwarz inequality)

For any two vectors $\vec{v}$ and $\vec{u}$ we have:

$$\boxed{-|\vec{u}|\,|\vec{v}| \le \vec{v}\cdot\vec{u} \le |\vec{u}|\,|\vec{v}|} \qquad (CS)$$

The equal sign holds if $\vec{u} = \gamma\vec{v}$ for $\gamma \in \mathbb{R}$.

Proof: define $\gamma = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}$

$$0 \le |\vec{u} - \gamma\vec{v}|^2 \qquad (*)$$

$$= (\vec{u} - \gamma\vec{v})\cdot(\vec{u} - \gamma\vec{v}) = |\vec{u}|^2 + \gamma^2|\vec{v}|^2 - 2\gamma\,\vec{u}\cdot\vec{v}$$

$$= |\vec{u}|^2 - \frac{(\vec{u}\cdot\vec{v})^2}{|\vec{v}|^2} - 2(\vec{u}\cdot\vec{v})^2/|\vec{v}|^2$$

$$= |\vec{u}|^2 - (\vec{u}\cdot\vec{v})^2/|\vec{v}|^2 \implies (\vec{u}\cdot\vec{v})^2 \le |\vec{u}|^2|\vec{v}|^2$$

$$\implies -|\vec{u}||\vec{v}| \le \vec{u}\cdot\vec{v} \le |\vec{u}||\vec{v}|$$

- equality for $\vec{u} = \gamma\vec{v}$ from $(*)$

Proposition: (Triangle Inequality)

For any two vectors $\vec{u}$ and $\vec{v}$ we have

$$\boxed{|\vec{u}+\vec{v}| \leq |\vec{u}| + |\vec{v}|} \qquad (TI)$$

The equal sign holds iff $\vec{u} = \gamma \vec{v}$ $\quad \gamma \in \mathbb{R}^+$

Proof: $\quad |\vec{u}+\vec{v}|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$

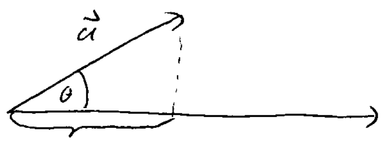
$$= |\vec{u}|^2 + |\vec{v}|^2 + 2\,\vec{u}\cdot\vec{v}$$

$$\underset{(CS)}{\leq} |\vec{u}|^2 + |\vec{v}|^2 + 2|\vec{u}||\vec{v}| = (|\vec{u}| + |\vec{v}|)^2$$

$$\underset{\text{take the square root}}{\Rightarrow} \quad |\vec{u}+\vec{v}| \leq |\vec{u}| + |\vec{v}|$$

- The equality holds for $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|$, i.e. if $\vec{u} = \gamma\vec{v}$ $\quad \gamma \in \mathbb{R}^+$ ∎

Def.: The <u>projection of a vector $\vec{u}$ onto $\vec{v}$</u> is the length of the line segment $\vec{u}\cdot\vec{e}_v$, where $\vec{e}_v$ is a unit vector in the direction of $\vec{v}$, i.e. $|\vec{e}_v| = 1$.



$$\vec{u}\cdot\vec{e}_v = |\vec{u}||\vec{e}_v|\cos\theta = |\vec{u}|\cos\theta$$

We can also write $\vec{e}_v = \frac{1}{|\vec{v}|}\vec{v}$. Clearly $\vec{e}_v\cdot\vec{e}_v = \frac{1}{|\vec{v}|^2}\vec{v}\cdot\vec{v} = 1$

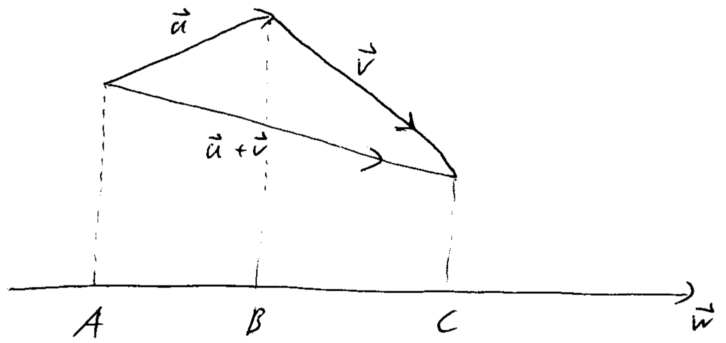
Expl 1) Determine the projection of the vector $\vec{u} = 4\vec{i} - 3\vec{j} + 5\vec{k}$ onto the vector $\vec{v} = 3\vec{i} + 5\vec{j} - 4\vec{k}$.

- first find the unit vector in the direction $\vec{v}$:

$$|\vec{v}|^2 = 3^2 + 25 + 16 = 50 \Rightarrow \vec{e}_v = \frac{1}{\sqrt{50}}\vec{v} \Rightarrow \vec{u}\cdot\vec{e}_v = \frac{1}{\sqrt{50}}(12-15-20) = \frac{-23}{5\sqrt{2}}$$

Expl 2) Proof of linearity of properties (P) in section 2.5.

$$(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w} \qquad (*)$$



projection of $(\vec{u}+\vec{v})$ onto $\vec{w}$; $(\vec{u}+\vec{v})\cdot\vec{e}_w = AC$

projection of $\vec{u}$ onto $\vec{w}$ + projection of $\vec{v}$ onto $\vec{w}$; $\vec{u}\cdot\vec{e}_w + \vec{v}\cdot\vec{e}_w = AB + BC$

Since $AC = AB + BC$

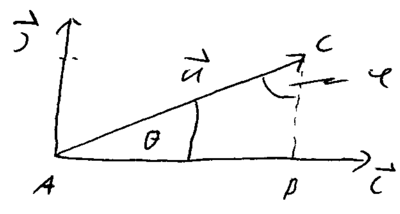
$$\Rightarrow (\vec{u}+\vec{v})\cdot\vec{e}_w = \vec{u}\cdot\vec{e}_w + \vec{v}\cdot\vec{e}_w$$

$$\Rightarrow (\vec{u}+\vec{v})\cdot\frac{\vec{w}}{|\vec{w}|} = \vec{u}\cdot\frac{\vec{w}}{|\vec{w}|} + \vec{v}\cdot\frac{\vec{w}}{|\vec{w}|} \quad |\cdot|\vec{w}| \quad \Rightarrow (*) \quad \blacksquare$$

Expl 3) Proof (using vectors) of the identity

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

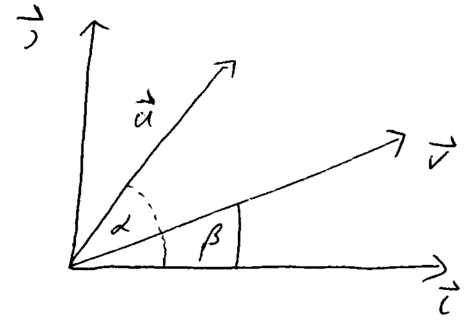
first note that



$$AB = |\vec{u}|\cos\theta \qquad \Rightarrow \quad \vec{AB} = |\vec{u}|\cos\theta\,\vec{i}$$

$$CB = |\vec{u}|\cos\varphi = |\vec{u}|\cos\left(\tfrac{\pi}{2}-\theta\right) = |\vec{u}|\sin\theta \Rightarrow \vec{BC} = |\vec{u}|\sin\theta\,\vec{j}$$

$$\Rightarrow \quad \vec{u} = \vec{AB} + \vec{BC} = |\vec{u}|\left(\cos\theta\,\vec{i} + \sin\theta\,\vec{j}\right).$$



$$\Rightarrow \quad \vec{u} = |\vec{u}| \, (\cos\alpha \, \vec{i} + \sin\alpha \, \vec{j})$$

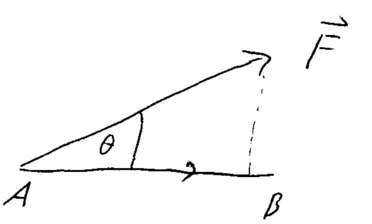
$$\vec{v} = |\vec{v}| \, (\cos\beta \, \vec{i} + \sin\beta \, \vec{j})$$

direct calculation: $|\vec{u}| \, |\vec{v}| \, (\cos\alpha \cos\beta + \sin\alpha \sin\beta) = \vec{u} \cdot \vec{v}$

from def. of dot product: $\vec{u} \cdot \vec{v} = |\vec{u}| \, |\vec{v}| \cos(\alpha - \beta)$

$$\Rightarrow \quad \cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta \quad \blacksquare$$

Expl 4)



$$\text{Work} = (\text{Force}) \cdot (\text{Displacement})$$

$$= \underbrace{|\vec{F}| \cos\theta}_{\text{component of } \vec{F} \text{ in the direction AB}} \, |\vec{AB}|$$

skipped → see also exercise

(we do not worry here about units, leave it to the physicists)

Find the work needed to move an object from a point $A(3, 2, -1)$ to $B(2, -1, 4)$ when a force $\vec{F} = 4\vec{i} - 3\vec{j} + 2\vec{k}$ is applied:

$$\text{vector } \vec{AB} = \vec{OB} - \vec{OA} = (2-3)\vec{i} + (-1-2)\vec{j} + (4+1)\vec{k}$$

$$= -\vec{i} - 3\vec{j} + 5\vec{k}$$

$$\text{work} = |\vec{F}| \cos\theta \, |\vec{AB}|$$

$$= \vec{F} \cdot \vec{AB} \cos\theta$$

$$= (4\vec{i} - 3\vec{j} + 2\vec{k}) \cdot (-\vec{i} - 3\vec{j} + 5\vec{k})$$

$$= -4 + 9 + 10 = \underline{\underline{15}}$$

2.7 The cross or vector product (orientation)

First let us see the notion of orientation in the plane:

We seek a vector $(\vec{v})^{\perp}$ which results when rotating the vector $\vec{v}$ by $\frac{\pi}{2}$. Direction?

Let us define this formally:

Def.: An <u>orientation</u> in a two dimensional Euclidean space S is an operation which assigns to each vector $\vec{v} \in S$ a vector $(\vec{v})^{\perp}$ with $(\lambda, \mu \in \mathbb{R})$

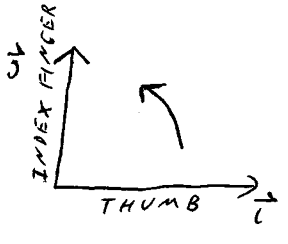
i) $(\lambda \vec{v} + \mu \vec{u})^{\perp} = \lambda (\vec{v})^{\perp} + \mu (\vec{u})^{\perp}$ (linear)

ii) $(\vec{v})^{\perp} \perp \vec{v} \equiv (\vec{v})^{\perp} \cdot \vec{v} = 0$ (perpendicular)

iii) If $|\vec{v}| = 1$ then also $|(\vec{v})^{\perp}| = 1$ (length preserving)

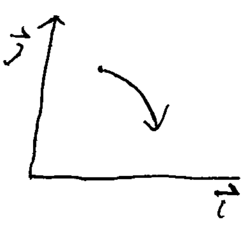
Expl.: Take an orthonormal basis $\vec{i}, \vec{j}$. Then there exist two possible choices to define an orientation

<u>right handed</u>



$(\vec{i})^{\perp} = +\vec{j}$ and $(\vec{j})^{\perp} = -\vec{i}$

<u>left handed</u>



$(\vec{i})^{\perp} = -\vec{j}$ and $(\vec{j})^{\perp} = +\vec{i}$

check if the properties of the definition hold right handed orientation

any vector $\vec{v} = x\vec{i} + y\vec{j}$

i) $(\vec{v})^{\perp} = x\vec{j} - y\vec{i}$

ii) $\vec{v} \cdot (\vec{v})^{\perp} = (x\vec{i} + y\vec{j}) \cdot (x\vec{j} - y\vec{i}) = -xy + xy = 0$

iii) $|\vec{v}|^2 = x^2 + y^2 \qquad |(\vec{v})^{\perp}|^2 = x^2 + y^2$

All properties hold.

- An example for a nonorientable surface is the Möbius band. A RHO comes back as a LHO when one moves once around the strip.

- Now 3D:

<u>Def.</u>: An <u>orientation</u> in a three dimensional space is an operation which assigns to each pair of vectors $\vec{u}$ and $\vec{v}$ a new vector

$$\boxed{\vec{w} = \vec{u} \times \vec{v}}$$

(read $\vec{u}$ cross $\vec{v}$) with

i) The map $(\vec{u}, \vec{v}) \to \vec{u} \times \vec{v}$ is bilinear

$$\boxed{\begin{aligned} (\lambda\vec{u} + \mu\vec{v}) \times \vec{w} &= \lambda\,\vec{u} \times \vec{w} + \mu\,\vec{v} \times \vec{w} \\ \vec{w} \times (\lambda\vec{u} + \mu\vec{v}) &= \lambda\,\vec{w} \times \vec{u} + \mu\,\vec{w} \times \vec{v} \end{aligned}} \qquad (C1)$$

ii) The vector $\vec{w} = \vec{u} \times \vec{v}$ is perpendicular to $\vec{u}$ and $\vec{v}$, i.e.

$$\boxed{\vec{w} \cdot \vec{u} = \vec{w} \cdot \vec{v} = 0} \qquad (C2)$$

iii) For any $\vec{v} \in S$

$$\boxed{\vec{v} \times \vec{v} = 0} \qquad (C3)$$

iv) If $\vec{u} \cdot \vec{v} = 0$, $|\vec{u}| = 1$, $|\vec{v}| = 1$ then $\boxed{|\vec{v} \times \vec{u}| = 1}$ (C4)

Consequences:

Proposition: The vector product is antisymmetric

$$\boxed{\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}} \qquad (C5)$$

Proof: $(\vec{u} + \vec{v}) \times (\vec{u} + \vec{v}) = \underbrace{\vec{u} \times \vec{u}}_{0 \leftarrow (C3)} + \underbrace{\vec{v} \times \vec{v}}_{(C3) \rightarrow 0} + \vec{u} \times \vec{v} + \vec{v} \times \vec{u} \underset{(C3)}{=} 0$

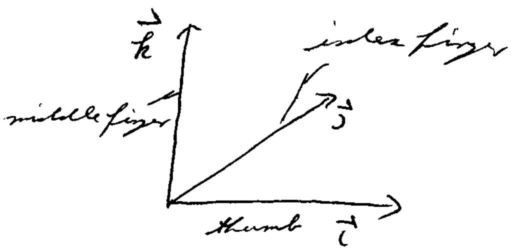
$\Rightarrow \quad \vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

Orthogonal basis: $\vec{i}, \vec{j}, \vec{k}$

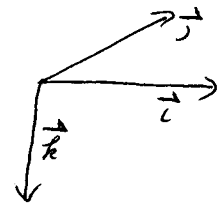
By (C4) there are only two possibilities either $\vec{i} \times \vec{j} = k$

or $\vec{i} \times \vec{j} = -k$

right handed



left handed



Consider right handed system and deduce all six possible identities

$\boxed{\vec{i} \times \vec{j} = \vec{k}} \underset{(C5)}{\Rightarrow} \boxed{\vec{j} \times \vec{i} = -\vec{k}}$

$\vec{i} \times \vec{k} = \pm \vec{j} \quad \Rightarrow \quad \vec{i} \times (\vec{j} + \vec{k}) \underset{(C1)}{=} \vec{k} \pm \vec{j} \quad \Rightarrow \quad \boxed{\vec{i} \times \vec{k} = -\vec{j}}$

$\underset{(C5)}{\Rightarrow} \boxed{\vec{k} \times \vec{i} = \vec{j}}$

$\vec{k} \times \vec{j} = \pm \vec{i} \quad \Rightarrow \quad (\vec{k} + \vec{i}) \times \vec{j} = \pm \vec{i} + \vec{k} \quad \Rightarrow \quad \boxed{\vec{k} \times \vec{j} = -\vec{i}}$

$\underset{(C5)}{\Rightarrow} \boxed{\vec{j} \times \vec{k} = \vec{i}}$

- We have plus signs when i, j, k are in cyclic order and minus signs when they are anti-cyclic.
- All signs are reversed in left handed systems. (exercise)

For two general vectors

$$\vec{u} = u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k} \qquad \vec{V} = V_1 \vec{i} + V_2 \vec{j} + V_3 \vec{k}$$

we have (in a right handed system)

$$\vec{u} \times \vec{V} = (u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k}) \times (V_1 \vec{i} + V_2 \vec{j} + V_3 \vec{k})$$

$$= (u_2 V_3 - u_3 V_2)\vec{i} + (u_3 V_1 - u_1 V_3)\vec{j} + (u_1 V_2 - u_2 V_1)\vec{k}$$

A convenient way to remember this is

$$\boxed{\vec{u} \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ V_1 & V_2 & V_3 \end{vmatrix}} \qquad (C6)$$

later $|\vec{u} \times \vec{v}| \equiv$ area of a parallelogram

Expl: Find a vector of magnitude 5 which is perpendicular to the plane which contains the vectors

$$\vec{V} = 2\vec{i} + \vec{j} - 3\vec{k} \qquad \vec{u} = \vec{i} - 2\vec{j} + \vec{k}.$$

- First find a vector perpendicular to the plane

$$\vec{W} = \vec{V} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -3 \\ 1 & -2 & 1 \end{vmatrix} = -5\vec{i} - 5\vec{j} - 5\vec{k}$$

Also $\vec{u} \times \vec{v} = -\vec{w}$ is a solution. $\Rightarrow$ $\vec{w} = \pm 5(\vec{i} + \vec{j} + \vec{k})$

magnitude: $|\vec{W}|^2 = \pm 3 \cdot 5^2$

$\Rightarrow$ The vectors $\pm \frac{5}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k})$ are perpendicular to the plane which contains the vectors $\vec{u}$ and $\vec{v}$.

check: $\pm \frac{5}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k}) \cdot (2\vec{i} + \vec{j} - 3\vec{k}) = \pm\frac{5}{\sqrt{3}}(2 + 1 - 3) = 0$ ✓

$\pm \frac{5}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k}) \cdot (\vec{i} - 2\vec{j} + \vec{k}) = \pm\frac{5}{\sqrt{3}}(1 - 2 + 1) = 0$ ✓

2.8) Triple products

Let's see now which kind of combinations we can form using the dot and the cross product.

i) In general we have

$$\underbrace{(\vec{u}\cdot\vec{v})}_{}\,\vec{w} \neq \vec{u}\,(\vec{v}\cdot\vec{w})$$

(scalar)(vector) (vector)(scalar) = (vector)

ii)

$$\underbrace{(\vec{u}\times\vec{v})}_{}\,\vec{w}\ ,\quad \vec{u}\,(\vec{v}\times\vec{w}) \quad \text{do not make sense!}$$

(vector) ? (vector) (vector) ? (vector)

iii) scalar triple product (volume of a parallelepiped)

$$\boxed{\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}} \in \mathbb{R} \text{ (from (C6))} \quad (TP1)$$

The following properties follow directly from the properties of the determinant (see algebra course for details):

a) $|*| = -|*|$ if a row or column is interchanged

$$\Rightarrow \vec{u}\cdot(\vec{v}\times\vec{w}) = -\vec{v}\cdot(\vec{u}\times\vec{w}) = \vec{w}\cdot(\vec{u}\times\vec{v}) = -\vec{w}\cdot(\vec{v}\times\vec{u})$$

b) $\vec{u}\cdot(\vec{v}\times\vec{w}) = 0$ if $\vec{u}$, $\vec{v}$ and $\vec{w}$ are linearly dependent
and "$\Leftarrow$" if $\vec{u}$, $\vec{v}$ and $\vec{w}$ are linearly dependent then
$\vec{u}\cdot(\vec{v}\times\vec{w}) = 0$ (recall 2.3)

- this follows from the fact that $|*| = 0$ if the rows or columns are linearly dependent.

iv) $\boxed{\vec{u} \times \vec{v} \times \vec{w} = (\vec{u} \cdot \vec{w})\,\vec{v} - (\vec{u} \cdot \vec{v})\,\vec{w}}$ ($\equiv$ vector!) (TP2)

Proof: Choose the direction of your orthogonal basis such that

$$\vec{u} = \gamma \vec{i} \qquad \gamma \in \mathbb{R}$$

$$\vec{u} \times \vec{v} \times \vec{w} = \vec{u} \times \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= \vec{u} \times \Big[\underbrace{(v_2 w_3 - w_2 v_3)}_{a}\,\vec{i} + \underbrace{(v_3 w_1 - v_1 w_3)}_{b}\,\vec{j} + \underbrace{(v_1 w_2 - v_2 w_1)}_{c}\,\vec{k}\Big]$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \gamma & 0 & 0 \\ a & b & c \end{vmatrix} = -\gamma c\,\vec{j} + \gamma b\,\vec{k}$$

$$= (-\gamma v_1 w_2 + \gamma v_2 w_1)\,\vec{j} + (\gamma v_3 w_1 - \gamma v_1 w_3)\,\vec{k}$$

$$= \gamma w_1 (v_2 \vec{j} + v_3 \vec{k}) - \gamma v_1 (w_2 \vec{j} + w_3 \vec{k}) + \underbrace{\gamma v_1 w_1 \vec{i} - \gamma v_1 w_1 \vec{i}}_{+0}$$

$$= \underbrace{\gamma w_1}_{\vec{u}\cdot\vec{w}} \underbrace{(v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k})}_{\vec{v}} - \underbrace{\gamma v_1}_{\vec{u}\cdot\vec{v}} \underbrace{(w_1 \vec{i} + w_2 \vec{j} + w_3 \vec{k})}_{\vec{w}}$$

$$= (\vec{u} \cdot \vec{w})\,\vec{v} - (\vec{u} \cdot \vec{v})\,\vec{w} \qquad \checkmark$$

2.9) Quadruple products

i) $\boxed{(\vec{u} \times \vec{v}) \cdot (\vec{w} \times \vec{x}) = (\vec{u} \cdot \vec{w})(\vec{v} \cdot \vec{x}) - (\vec{u} \cdot \vec{x})(\vec{v} \cdot \vec{w})}$ (QP1) $\in \mathbb{R}$

Proof: $(\vec{u} \times \vec{v}) \cdot (\vec{w} \times \vec{x}) = \vec{w} \cdot (\vec{x} \times \vec{u} \times \vec{v})$ by (TP1) a)

$$= \vec{w} \cdot \big[(\vec{x} \cdot \vec{v})\,\vec{u} - (\vec{x} \cdot \vec{u})\,\vec{v}\big]$$

$$= (\vec{x} \cdot \vec{v})(\vec{w} \cdot \vec{u}) - (\vec{x} \cdot \vec{u})(\vec{w} \cdot \vec{v}) \qquad \checkmark$$

We can use this identity to show:

$$\boxed{|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}| \sin\theta} \qquad (QP2)$$

for any two vectors with θ being the angle between them.

Proof:

$$|\vec{u} \times \vec{v}|^2 = (\vec{u} \times \vec{v}) \cdot (\vec{u} \times \vec{v})$$

$$= (\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v}) - (\vec{u} \cdot \vec{v})(\vec{v} \cdot \vec{u})$$

$$\overrightarrow{(QP1)} \quad = |\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2 = |\vec{u}|^2 |\vec{v}|^2 - |\vec{u}|^2 |\vec{v}|^2 \cos^2\theta$$

$$= |\vec{u}|^2 |\vec{v}|^2 \underbrace{(1 - \cos^2\theta)}_{\sin^2\theta}$$

take the square root $\Rightarrow$ (QP2)

ii)

$$\boxed{\vec{u} \times \vec{v} \times \vec{w} \times \vec{x} = [\vec{u} \cdot (\vec{v} \times \vec{x})]\,\vec{w} - [\vec{u} \cdot (\vec{v} \times \vec{w})]\,\vec{x}} \qquad (QP3)$$

(vector!)

Proof:

$$(\vec{u} \times \vec{v}) \times \vec{w} \times \vec{x} = [\underbrace{(\vec{u} \times \vec{v}) \cdot \vec{x}}_{\vec{u} \cdot (\vec{v} \times \vec{x})}]\,\vec{w} - [\underbrace{(\vec{u} \times \vec{v}) \cdot \vec{w}}_{\vec{u} \cdot (\vec{v} \times \vec{w})}]\,\vec{x} \quad \text{by (TP2)}$$

Consequence: the vectors $\vec{u}$, $\vec{v}$, $\vec{w}$, $\vec{x}$ are linearly dependent, that is

$$\vec{x} = a_1 \vec{u} + a_2 \vec{v} + a_3 \vec{w} \qquad a_1, a_2, a_3 \in \mathbb{R}$$

(see section 2.3)

Proof:

$$(\vec{u} \times \vec{v}) \times (\vec{w} \times \vec{x}) = -(\vec{w} \times \vec{x}) \times (\vec{u} \times \vec{v})$$

$$= -[\vec{w} \cdot (\vec{x} \times \vec{v})]\,\vec{u} + [\vec{w} \cdot (\vec{x} \times \vec{u})]\,\vec{v}$$

$$\Rightarrow [\vec{u} \cdot (\vec{v} \times \vec{x})]\,\vec{w} - [\vec{u} \cdot (\vec{v} \times \vec{w})]\,\vec{x} = [\vec{w} \cdot (\vec{x} \times \vec{u})]\,\vec{v} - [\vec{w} \cdot (\vec{x} \times \vec{v})]\,\vec{u}$$

$$\Rightarrow \vec{x} = \frac{\vec{w} \cdot (\vec{x} \times \vec{v})}{\vec{u} \cdot (\vec{v} \times \vec{w})}\,\vec{u} - \frac{\vec{w} \cdot (\vec{x} \times \vec{u})}{\vec{u} \cdot (\vec{v} \times \vec{w})}\,\vec{v} + \frac{\vec{u} \cdot (\vec{v} \times \vec{x})}{\vec{u} \cdot (\vec{v} \times \vec{w})}\,\vec{w}$$

2.10) Vectorfields

Def.: A *vectorfield* on $U \subset \mathbb{R}^n$ is a map which associates to each point $P \in U$ a vector $\vec{F}(P) \in \mathbb{R}^n$, that is $P \to \vec{F}(P)$.

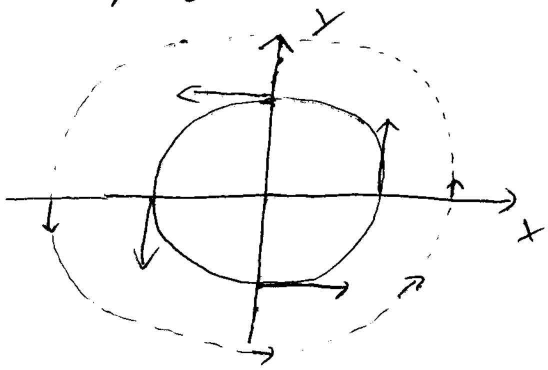
In an orthogonal basis (e.g. $n=3$):

$$\vec{F}(P) = f_1(P)\,\vec{i} + f_2(P)\,\vec{j} + f_3(P)\,\vec{k}$$

A vectorfield is said to be differentiable if the component functions, i.e. f_1, f_2, f_3, are differentiable.

Expl 1) Vortexfield $(n=2)$

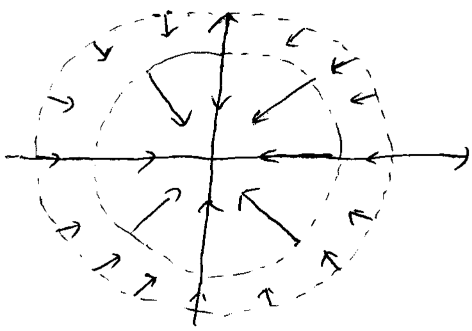
$$\vec{F}(x,y) = \frac{1}{r^2}\left(-y\,\vec{i} + x\,\vec{j}\right) \qquad r^2 = x^2 + y^2$$



$U = \mathbb{R}^2 \setminus (0,0)$

Expl 2) Gravitational field $(n=3)$

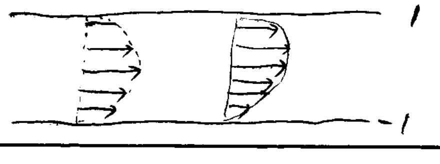
$$\vec{G}(x,y,z) = \frac{-1}{r^3}\left(x\,\vec{i} + y\,\vec{j} + z\,\vec{k}\right) \qquad r^2 = x^2 + y^2 + z^2$$



$U = \mathbb{R}^3 \setminus (0,0,0)$

Expl 3) Current field (velocity in a moving fluid) $(n=2)$

$\vec{H}(x,y) = (1-y^2)\,\vec{i}$ $\qquad U = \mathbb{R} \times [-1,1]$



Expl 4/ The gradient

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\,\vec{i} + \frac{\partial f}{\partial y}\,\vec{j} + \frac{\partial f}{\partial z}\,\vec{k}$$

The gradient points into the direction in which f is changing most rapidly.

Further important notions of a vectorfield are its <u>divergence</u>

$$\nabla \cdot \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial x} + \frac{\partial f_3}{\partial x}$$

and its <u>rotation</u>

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix} .$$

We will not discuss these quantities further in this course.