4) Analytic Geometry (3D)

4.1) Points

Now a point in $\mathbb{R}^3$ is uniquely identified by three coordinates. For example

$$P(x_0, y_0, z_0) \qquad \text{or} \qquad P(\theta, \phi, r)$$

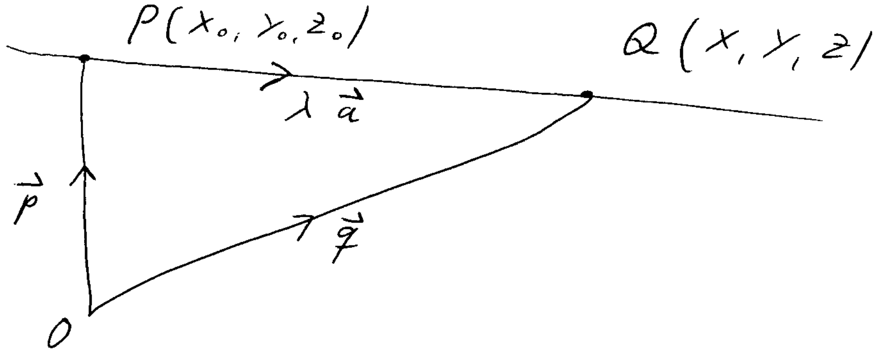
The distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is now given by

$$\boxed{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

(in Cartesian coordinates)

4.2) Lines

We know that we should be able to construct a line from two points. Consider



- $\vec{a} = (a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k})$ is a vector in the direction $\overrightarrow{PQ}$
- with $\lambda \in \mathbb{R}$ we can always achieve that $\lambda \vec{a} = \overrightarrow{PQ}$
- $\vec{p} = x_0 \vec{i} + y_0 \vec{j} + z_0 \vec{k} \qquad \vec{q} = x \vec{i} + y \vec{j} + z \vec{k}$

$$\Rightarrow \qquad \vec{q} = \vec{p} + \lambda \vec{a} \qquad \text{(vector equation)}$$

$$x \vec{i} + y \vec{j} + z \vec{k} = x_0 \vec{i} + y_0 \vec{j} + z_0 \vec{k} + \lambda (a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k})$$

$\because$ $\vec{i}, \vec{j}, \vec{k}$ are linearly independent

$$\Rightarrow \quad x = x_0 + \lambda a_1 \qquad y = y_0 + \lambda a_2 \qquad z = z_0 + \lambda a_3$$

$$\Rightarrow \quad Q(x, y, z) = Q(x_0 + \lambda a_1,\ y_0 + \lambda a_2,\ z_0 + \lambda a_3)$$

$\Rightarrow$ The equation of a line through the point $P(x_0, y_0, z_0)$ parallel to the vector $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ can be written as

$$\boxed{\frac{x - x_0}{a_1} = \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3} = \lambda}$$ Cartesian form

Expl.: Find the equation of the line in Cartesian form through the point $P(1, 3, -1)$ parallel to the vector $\vec{a} = 2\vec{i} + 3\vec{j} + 4\vec{k}$. Characterise each point on the line:

$$\Rightarrow \quad \frac{x-1}{2} = \frac{y-3}{3} = \frac{z+1}{4} = \lambda$$

Each point on the line can be expressed as

$$Q(x, y, z) = Q(2\lambda + 1,\ 3\lambda + 3,\ 4\lambda - 1)$$

4.2.1) Intersection of lines

Two lines can either intersect in one point or they are parallel or skew. Thus

$$L_1 : \quad \frac{x - x_1}{a_1} = \frac{y - y_1}{a_2} = \frac{z - z_1}{3} = \lambda$$

$$L_2 : \quad \frac{x - x_2}{b_1} = \frac{y - y_2}{b_2} = \frac{z - z_2}{b_3} = \mu$$

intersect iff $Q(\lambda a_1 + x_1,\ \lambda a_2 + y_1,\ \lambda a_3 + z_1)$ equals

$P(\mu b_1 + x_2, \mu b_2 + y_2, \mu b_3 + z_2)$. This means one has to solve

$$\lambda a_1 + x_1 = \mu b_1 + x_2$$
$$\lambda a_2 + y_1 = \mu b_2 + y_2$$
$$\lambda a_3 + z_1 = \mu b_3 + z_2$$

for μ and λ.

Expl.: Does the line through the points $P(7,4,3)$ and $Q(9,5,5)$ intersect with the line of the previous example $\left(\frac{x-1}{2} = \frac{y-3}{3} = \frac{z+1}{4} = \lambda\right)$. In case they do, determine their point of intersection.

- First find a vector parallel to $\overline{PQ}$:

$$\vec{a} = \overrightarrow{PQ} = 2\vec{i} + \vec{j} + 2\vec{k}$$

we can now take either the point P or Q to determine the equation in Cartesian form:

$$\mathcal{L}_2: \quad \frac{x-7}{2} = \frac{y-4}{1} = \frac{z-3}{+2} = \mu$$

- Any point on $\mathcal{L}_2$ is of the form

$$R(2\mu + 7, \mu + 4, +2\mu + 3)$$

$\Rightarrow$ condition for intersection:

$$2\lambda + 1 = 2\mu + 7 \qquad (1)$$
$$3\lambda + 3 = \mu + 4 \qquad (2)$$
$$4\lambda - 1 = +2\mu + 3 \qquad (3)$$

from (1): $\lambda = \mu + 3$

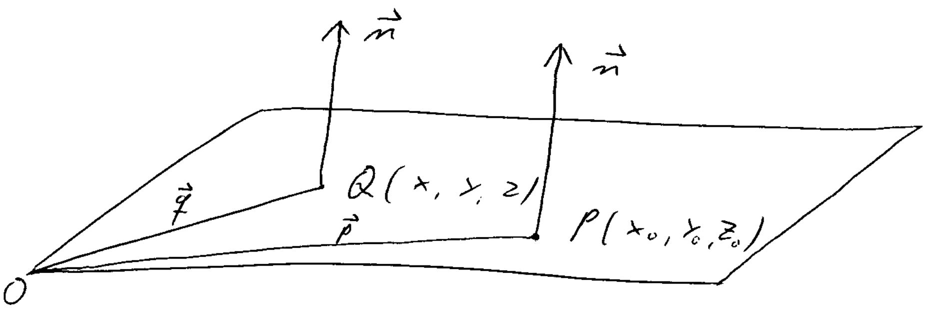
into (2): $3\mu + 9 + 3 = \mu + 4 \Rightarrow 2\mu = -8 \Rightarrow \underline{\mu = -4} \overset{(1)}{\Rightarrow} \underline{\lambda = -1}$

check (3): $4(-1) - 1 = (+2)(-4) + 3 \checkmark \Rightarrow$ point of intersection $Q(-1,0,-5)$

4.3) Planes

4.3.1) The equation of a plane

Any point Q in a plane P can be specified uniquely if we know one point $P(x_0, y_0, z_0)$ in the plane together with a normal vector $\vec{n}$ to the plane. (A normal vector is a vector such that $\vec{n} \cdot \vec{v} = 0$ for any vector in the plane.) Consider



$$\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$$

$$\vec{p} = x_0\vec{i} + y_0\vec{j} + z_0\vec{k}$$

$$\vec{q} = x\vec{i} + y\vec{j} + z\vec{k}$$

per definition: $(\vec{q} - \vec{p}) \cdot \vec{n} = 0$

$$\Rightarrow \boxed{(x - x_0)a + (y - y_0)b + (z - z_0)c = 0}$$

or $xa + by + cz + d = 0$

with $d = -(x_0 a + y_0 b + z_0 c)$

Expl.: Find the equation of the plane passing through the point $P(2, 4, -6)$ which is normal to the vector $\vec{n} = 4\vec{i} - \vec{j} + \vec{k}$.

$$(\vec{q} - \vec{OP}) \cdot \vec{n} = 0 \Leftrightarrow 4(x-2) - (y-4) + (z+6) = 0$$

$$\Leftrightarrow 4x - y + z + 2 = 0$$

equation of the plane from three points

From axiom 2 we know that there is a unique plane passing through three points. Say

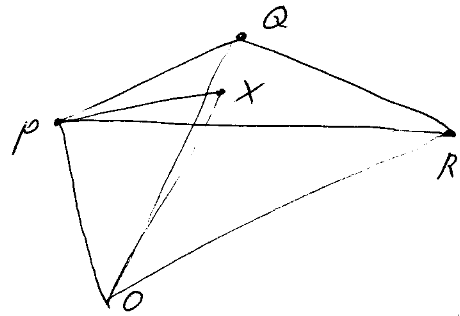
$$P(x_1, y_1, z_1) \qquad Q(x_2, y_2, z_2) \qquad R(x_3, y_3, z_3)$$

- The vectors $\vec{PQ}$ and $\vec{PR}$ are in the plane
- We need to construct a normal vector:

  The vector $\vec{PQ} \times \vec{PR}$ is perpendicular to the plane
- Now take an arbitrary point $X(x, y, z)$ in the plane.

  Then $\vec{RX}$ is in the plane and the equation

$$\vec{RX} \cdot (\vec{PQ} \times \vec{PR}) = 0$$

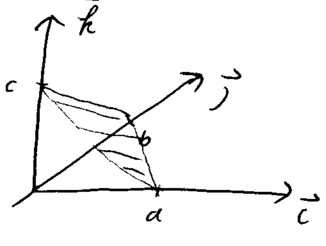


has to hold.

$$\Rightarrow \quad \begin{vmatrix} x - x_3 & y - y_3 & z - z_3 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

(see section 2.8 (iii) on the triple product)

Expl.: Find the equation of the plane through the three points $P(a, 0, 0)$, $Q(0, b, 0)$, $R(0, 0, c)$



$$\vec{PQ} = b\vec{j} - a\vec{i}$$

$$\vec{PR} = c\vec{k} - a\vec{i}$$

$$\vec{PX} = (x-a)\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = bc\,\vec{i} + ac\,\vec{j} + ab\,\vec{k}$$

$$\Rightarrow \vec{PX} \cdot (\vec{PQ} \times \vec{PR}) = (x-a)bc + acy + abz = 0 \iff \underline{\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1}$$

4.3.2) Intersection of a plane and a line

A plane $\mathcal{P}$ and a line $\mathcal{L}$ can intersect in a point P. To find P we have to solve simultaneously the equations

$$\mathcal{P}: \quad ax + by + cz + d = 0$$

$$\mathcal{L}: \quad \frac{x - x_0}{a_1} = \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3} = \lambda$$

for λ. Any point $Q \in \mathcal{L}$ can be written as $Q(a_1\lambda + x_0,\ a_2\lambda + y_0,\ a_3\lambda + z_0)$. For $Q \in \mathcal{P}$ we have to solve

$$a(a_1\lambda + x_0) + b(a_2\lambda + y_0) + c(a_3\lambda + z_0) + d = 0$$

$$\Rightarrow \lambda(aa_1 + ba_2 + ca_3) = -d - ax_0 - by_0 - cz_0$$

$$\Rightarrow \quad \lambda = -\frac{d + ax_0 + by_0 + cz_0}{aa_1 + ba_2 + ca_3} =: \tilde{\lambda}$$

Substituting this back into Q gives the point of intersection:

$$\boxed{Q(a_1\tilde{\lambda} + x_0,\ a_2\tilde{\lambda} + y_0,\ a_3\tilde{\lambda} + z_0) = \mathcal{L} \cap \mathcal{P}}$$

Expl.: Find the point of intersection of the line through the point $P(1, 3, 2)$ and $Q(2, 1, 1)$ with the plane $\mathcal{P}: 2x + y - z = 11$.

- First find $\mathcal{L}$ through P and Q:

$$\overrightarrow{PQ} = \vec{i} - 2\vec{j} - \vec{k}$$

$$\Rightarrow \quad \mathcal{L}: \quad \frac{x-1}{1} = \frac{y-3}{-2} = \frac{z-2}{-1} = \lambda$$

$$\Rightarrow \quad R(\lambda + 1,\ -2\lambda + 3,\ -\lambda + 2) \in \mathcal{L}$$

$$\Rightarrow \quad \text{if } R \in \mathcal{P}: \ 2(\lambda + 1) + (3 - 2\lambda) - (2 - \lambda) = 11$$

$$\Rightarrow 2\lambda - 2\lambda + \lambda + 2 + 3 - 2 - 11 = 0$$

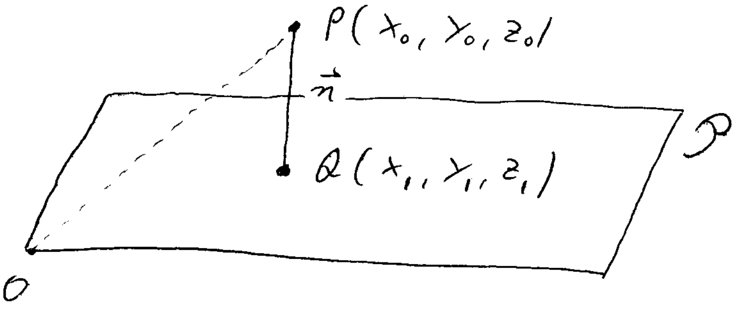
$$\Rightarrow \underline{\lambda = 8}$$

$$\Rightarrow \underline{R(9, -13, -6) \in \mathcal{L} \cap \mathcal{P}}$$

4.3.3) Shortest distance of a point P from a plane $\mathcal{P}$

What is the shortest (perpendicular) distance of a point $P(x_0, y_0, z_0)$ from the plane $\mathcal{P}$: $ax + by + cz + d = 0$?

Consider:



$Q \in \mathcal{P}$

- A normal vector to the plane is $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$
- The line $\overleftrightarrow{PQ}$ can be parameterised as

$$\overleftrightarrow{PQ} = \overrightarrow{OP} + \lambda \vec{n}$$

$\Rightarrow$ Any point R on this line is of the form

$$R(x_0 + \lambda a, y_0 + \lambda b, z_0 + \lambda c)$$

- For some particular value of λ, say $\lambda = \lambda_1$, we have $R = Q$.

$$\Rightarrow x_1 = x_0 + \lambda_1 a \quad ; \quad y_1 = y_0 + \lambda_1 b \quad ; \quad z_1 = z_0 + \lambda_1 c$$

- $\because Q \in \mathcal{P}$ it holds

$$a x_1 + b y_1 + c z_1 + d = 0$$

$$\Leftrightarrow a(x_0 + \lambda_1 a) + b(y_0 + \lambda_1 b) + c(z_0 + \lambda_1 c) + d = 0$$

$$\Leftrightarrow \lambda_1 (a^2 + b^2 + c^2) + a x_0 + b y_0 + c z_0 + d = 0$$

$$\Rightarrow \underline{\lambda_1 = -\frac{a x_0 + b y_0 + c z_0 + d}{a^2 + b^2 + c^2}}$$

Now we can compute $\overline{PQ} = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}$

$$= \sqrt{\lambda_1^2 (a^2 + b^2 + c^2)}$$

$$\Rightarrow \quad \boxed{\overline{PQ} = \left| \frac{a x_0 + b y_0 + c z_0 + d}{\sqrt{a^2 + b^2 + c^2}} \right|}$$

Expl.: Verify that the two lines

$$\mathcal{L}_1: \frac{x-4}{1} = \frac{y-1}{-1} = \frac{z+12}{-4} = \lambda \qquad \mathcal{L}_2: \frac{x+2}{-1} = \frac{y-11}{2} = \frac{z}{1} = \mu$$

intersect and find the equation of the plane containing both of them. Determine the shortest distance of the point $P(0,1,3)$ from this plane.

- Point on both lines are of the form

$$Q(\lambda+4, -\lambda+1, -4\lambda-12) \in \mathcal{L}_1 \qquad R(-\mu-2, 2\mu+11, \mu) \in \mathcal{L}_2$$

- For intersection we require $Q = R$:

$$\lambda + 4 = -\mu - 2 \qquad \Rightarrow \quad \lambda = -\mu - 6$$

$$1 - \lambda = 2\mu + 11 \qquad \Rightarrow \quad 1 + \mu + 6 = 2\mu + 11 \Rightarrow \underline{\mu = -4}$$

$$-4\lambda - 12 = \mu \qquad \Leftrightarrow \quad 8 - 12 = -4 \checkmark \qquad \Rightarrow \underline{\lambda = -2}$$

$\Rightarrow$ The point of intersection is $S(2, 3, -4)$.

- Now construct the equation of the plane:
  - The vectors $\vec{v} = \vec{i} - \vec{j} - 4\vec{k}$ and $\vec{w} = -\vec{i} + 2\vec{j} + \vec{k}$ are both in the plane.

$$\Rightarrow \quad \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & -4 \\ -1 & 2 & 1 \end{vmatrix} = 7\vec{i} + 3\vec{j} + \vec{k} \text{ is perpendicular to } \mathcal{P}.$$

$\Rightarrow \quad 7x + 3y + z + d = 0$ contains $\mathcal{L}_1$ and $\mathcal{L}_2$

$\because S \in \mathcal{P} \qquad 7 \cdot 2 + 3 \cdot 3 - 4 + d = 0 \qquad \Rightarrow \quad \underline{d = -19}$

$$\Rightarrow \quad \mathcal{P}: \; 7x + 3y + z = 19$$

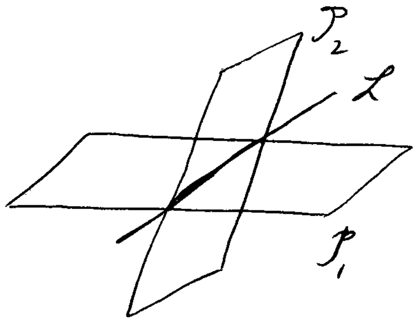
- Distance of $P(0,1,3)$ from the plane

$$d = \left| \frac{a x_0 + b y_0 + c z_0 + d}{\sqrt{a^2 + b^2 + c^2}} \right| = \left| \frac{3 \cdot 1 + 1 \cdot 3 - 19}{\sqrt{7^2 + 3^2 + 1}} \right| = \frac{13}{\sqrt{59}}$$

4.3.4/ The line of intersection of two planes

From axiom 6 we know that the intersection of two planes is a line.

$$\mathcal{P}_1 : \; a_1 x + b_1 y + c_1 z + d_1 = 0$$



$$\mathcal{P}_2 : \; a_2 x + b_2 y + c_2 z + d_2 = 0$$

- The vector $\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k} = \vec{n}_1 \times \vec{n}_2$ is parallel to the line L of intersection.

- We need also one point on the line. Take for instance $z = 0$ in $\mathcal{P}_1$ and $\mathcal{P}_2$ and add the equations of the planes:

$$(a_1 + a_2) x_1 + (b_1 + b_2) y_1 = -(d_1 + d_2) \qquad a_1 + a_2 \neq 0$$

$$\Rightarrow \quad x_1 = -\frac{b_1 + b_2}{a_1 + a_2} y_1 - \frac{d_1 + d_2}{a_1 + d_2}$$

$$\Rightarrow \quad -\frac{a_1 (b_1 + b_2)}{(a_1 + a_2)} y_1 - \frac{a_1 (d_1 + d_2)}{(a_1 + a_2)} + b_1 y_1 + d_1 = 0$$

$$\Rightarrow \quad y_1 \underbrace{\left(b_1 - \frac{a_1 (b_1 + b_2)}{a_1 + a_2} \right)}_{\frac{\cancel{a_1 b_1} + a_2 b_1 - \cancel{a_1 b_1} - a_1 b_2}{a_1 + d_2}} = \frac{a_1 (d_1 + d_2)}{a_1 + a_2} - d_1 = \frac{a_1 d_2 - d_1 a_2}{a_1 + a_2}$$

$$\Rightarrow \quad y_1 = \frac{a_1 d_2 - d_1 a_2}{a_2 b_1 - a_1 b_2} \qquad a_2 b_1 \neq a_1 b_2$$

interchange $a \Leftrightarrow b \Rightarrow x$

$$x_1 = \frac{b_1 d_2 - d_1 b_2}{b_2 a_1 - b_1 a_2}$$

$\Rightarrow$ The equation of the line is $\mathcal{L}: \ \frac{x - x_1}{V_1} = \frac{y - y_1}{V_2} = \frac{z}{V_3} = \lambda$
(Possibly you have to set x or y to zero if $a_2 b_1 = a_1 b_2$ or $a_1 = -a_2$)
Expl.: $\quad P_1: \ x + y + z - 5 = 0 \qquad P_2: \ 4x + y + 2z - 15 = 0$

$$\Rightarrow \quad \vec{n}_1 = \vec{i} + \vec{j} + \vec{k} \qquad \vec{n}_2 = 4\vec{i} + \vec{j} + 2\vec{k}$$

$$\Rightarrow \quad \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 4 & 1 & 2 \end{vmatrix} = \vec{i} + 2\vec{j} - 3\vec{k}$$

- point on the line: $\quad x_1 = \frac{1 \cdot (-15) - (-5) 1}{1 - 4} = \frac{10}{3}$

$$y_1 = \frac{1 \cdot (-15) - (-5) \cdot 4}{4 \cdot 1 - 1} = \frac{5}{3}$$

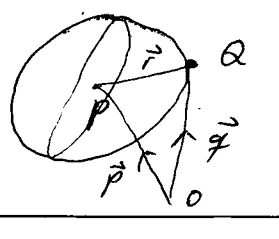
$\Rightarrow P\left(\frac{10}{3}, \frac{5}{3}, 0\right)$ is on the line $\mathcal{L}$

$$\Rightarrow \quad \mathcal{L}: \ \frac{x - \frac{10}{3}}{1} = \frac{y - \frac{5}{3}}{2} = \frac{z}{-3} = \lambda$$

4.4) The Sphere

4.4.1) The equation of the sphere

Taking a point $P(x_0, y_0, z_0)$, a sphere is the collection of all points $Q(x, y, z)$ which have the same distance r from this point. r is the radius of the sphere. According to section 4.1) (see formula for the distance) the equation for the sphere is therefore:



$$|\vec{p} - \vec{q}| = |\vec{r}| = r$$

In general, the equation of the sphere is therefore:

$$f(x,y) = x^2 + y^2 + z^2 - 2xx_0 - 2yy_0 - 2z_0 z - r_0^2 = 0$$

$$\Rightarrow \boxed{\mathcal{S}: \qquad (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = x_0^2 + y_0^2 + z_0^2 + r_0^2}$$

is a sphere with centre $P(x_0, y_0, z_0)$ and radius $r = \sqrt{x_0^2 + y_0^2 + z_0^2 + r_0^2}$

In particular if $P = O$: $x^2 + y^2 + z^2 = r_0^2$

Expl.: i) Find the centre and the radius of the sphere

$$\mathcal{S}: \qquad x^2 + y^2 + z^2 - 2x - 4y - 6z - 2 = 0$$

ii) Show that the intersection of this sphere and the plane

$$\mathcal{P}: \qquad x + 2y + 2z - 20 = 0$$

is a circle with centre $P(2, 4, 5)$

iii) Find the radius of the circle

Solⁿ.: i) Rewrite the equation for $\mathcal{S}$

$$(x^2 - 2x \underbrace{+1) - 1}_{0} + (y^2 - 4y \underbrace{+4) - 4}_{0} + (z^2 - 6z \underbrace{+9) - 9}_{0} - 2 = 0$$

$$\Leftrightarrow \quad (x-1)^2 + (y-2)^2 + (z-3)^2 = 16$$

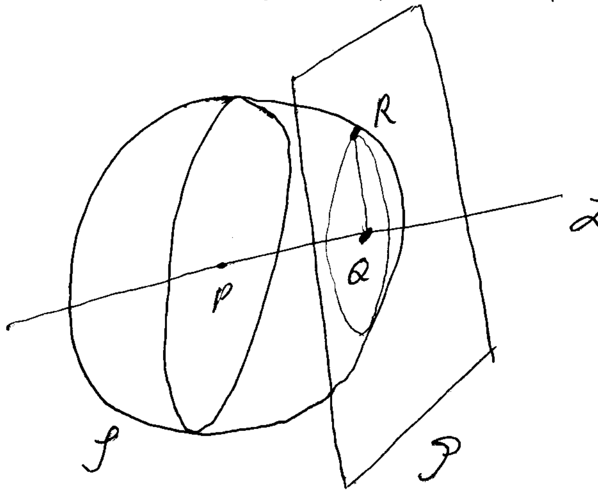
$\equiv$ sphere with radius 4 and centre $P(1, 2, 3)$

ii) The normal vector to $\mathcal{P}$ is

$$\vec{n} = \vec{i} + 2\vec{j} + 2\vec{k}$$

$\Rightarrow$ The equation of the line parallel to $\vec{n}$ passing through the centre is $\dfrac{x-1}{1} = \dfrac{y-2}{2} = \dfrac{z-3}{2} = \lambda$

$$\Rightarrow \quad Q(\lambda+1, 2\lambda+2, 2\lambda+3) \in \mathcal{P}$$



$$\Rightarrow (\lambda+1) + 2(2\lambda+2) + 2(2\lambda+3) - 20 = 0$$
$$\Leftrightarrow \lambda + 1 + 4\lambda + 4 + 4\lambda + 6 - 20 = 0$$
$$\Leftrightarrow 9\lambda - 9 = 0$$
$$\Rightarrow \lambda = 1$$
$$\Rightarrow \underline{Q = (2, 4, 5)}$$

iii) Inspect the picture

$$\Rightarrow \quad \overline{PR}^2 = \overline{PQ}^2 + \overline{QR}^2 \qquad \overline{PR} = r$$

$$\overrightarrow{QR} = \vec{i} + 2\vec{j} + 2\vec{k}$$

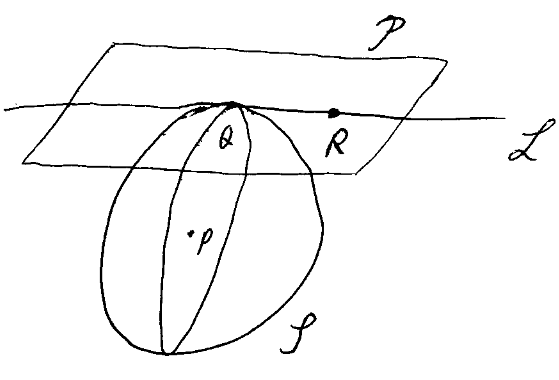
$$\Rightarrow \quad r^2 = 9 + |\overline{QR}|^2$$

$$\Rightarrow |\overline{QR}|^2 = 1 + 4 + 4 = 9$$

$\Rightarrow \quad \underline{\overline{QR} = \sqrt{7}}$ is the radius of the circle.

4.2.2) The tangent plane to the sphere

For a point $Q \in \mathcal{S}$ with centre P all lines through Q perpendicular to $\overline{PQ}$ lie in a plane called the tangent plane to the sphere.



$$Q \in \mathcal{P} \quad \wedge \quad Q \in \mathcal{S}$$

$$\Rightarrow \text{if } R \in \mathcal{P} \quad \Rightarrow \quad \overrightarrow{PQ} \cdot \overrightarrow{QR} = 0$$

Now specify: $P(x_0, y_0, z_0)$, $Q(x_1, y_1, z_1)$, $R(x, y, z)$

$\Rightarrow \quad \vec{PQ} = (x_1 - x_0)\vec{i} + (y_1 - y_0)\vec{j} + (z_1 - z_0)\vec{k}$

$\quad\;\; \vec{QR} = (x - x_1)\vec{i} + (y - y_1)\vec{j} + (z - z_1)\vec{k}$

$\Rightarrow$ The equation of the tangent plane is therefore

$$\vec{PQ} \cdot \vec{QR} = (x_1 - x_0)(x - x_1) + (y_1 - y_0)(y - y_1) + (z_1 - z_0)(z - z_1) = 0$$

Expl.: Find the equation of the sphere with centre $P(5, -10, 5)$ which touches the plane

$$\mathcal{P}: \; 9x + 12y + 20z = 0$$

Sol^n: The distance of P from the plane $\mathcal{P}$ is (see 4.3.3)

$$d = \left| \frac{a x_0 + b y_0 + c z_0 + d}{\sqrt{a^2 + b^2 + c^2}} \right| = \left| \frac{9 \cdot 5 - 12 \cdot 10 + 20 \cdot 5}{\sqrt{9^2 + 12^2 + 20^2}} \right|$$

$$= \left| \frac{45 - 120 + 100}{\sqrt{\underset{81}{\cancel{45}} + 144 + 400}} \right| = \frac{25}{25} = 1 \equiv \text{radius of } \mathcal{S}$$

$$\Rightarrow \quad \mathcal{S}: \; (x-5)^2 + (y+10)^2 + (z-5)^2 = 1$$