Coursework 1

1) Suppose $\mathcal{L}_1$ is the intersection of $\mathcal{P}_2$ and $\mathcal{P}_3$, i.e. $\mathcal{L}_1 = \mathcal{P}_2 \cap \mathcal{P}_3$.

(10) Also $\mathcal{L}_2 = \mathcal{P}_1 \cap \mathcal{P}_3$; $\mathcal{L}_3 = \mathcal{P}_1 \cap \mathcal{P}_2$

$\because$ $\mathcal{L}_1$ and $\mathcal{L}_2$ are both in $\mathcal{P}_3$ they are either parallel or cross

- suppose they cross in a point P

$$\Rightarrow \quad P \in \mathcal{P}_2 \quad \because \quad \mathcal{L}_1 \in \mathcal{P}_2$$
$$\Rightarrow \quad P \in \mathcal{P}_3 \quad \because \quad \mathcal{L}_1 \in \mathcal{P}_3$$
$$\Rightarrow \quad P \in \mathcal{P}_1 \quad \because \quad \mathcal{L}_2 \in \mathcal{P}_1$$
$$\Rightarrow \quad P \in \mathcal{L}_3 \quad \because \quad \mathcal{L}_3 = \mathcal{P}_1 \cap \mathcal{P}_2$$

$\Rightarrow$ All three lines cross in P

- suppose now they do not cross

$\because$ $\mathcal{L}_1$ and $\mathcal{L}_2$ are coplanar they cross or are parallel

- from the previous case follows that $\mathcal{L}_3$ can not cross $\mathcal{L}_1$ as otherwise there were points common to $\mathcal{L}_1\ \mathcal{L}_2\ \mathcal{L}_3$

$\because$ $\mathcal{L}_1$ and $\mathcal{L}_3$ are coplanar and do not cross, they are parallel

- similar argument to show $\mathcal{L}_1$ is parallel to $\mathcal{L}_3$

2) $|\vec{u}| = \sqrt{52 + k^2} \qquad |\vec{v}| = \sqrt{85 + k^2/4} \qquad \vec{u}\cdot\vec{v} = 54$

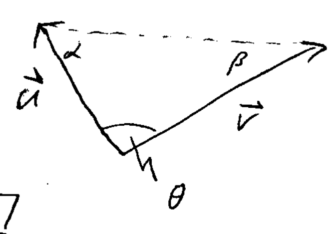
(10)

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|} = \frac{54}{\sqrt{(52+k^2)(85+k^2/4)}} \Rightarrow k = \pm 2\sqrt{9\sqrt{34} - 49}$$

3)

i) $|\vec{u}| = 1 \quad |\vec{v}| = 1 \qquad \vec{u}\cdot\vec{v} = -\frac{1}{2}\tau_+ \Rightarrow \cos\theta = -\frac{1}{2}\tau_+ \Rightarrow \theta = \frac{4}{5}\pi = 144°$

(25)



$$\cos\alpha = \frac{(\vec{v}-\vec{u})\cdot(-\vec{u})}{|\vec{u}|\,|\vec{v}-\vec{u}|} \quad ; \quad |\vec{v}-\vec{u}| = \sqrt{\tfrac{1}{2}(5+\sqrt{5})}$$

$\boxed{12}$

$$(\vec{v}-\vec{u})\cdot(-\vec{u}) = \frac{1}{4}(5+\sqrt{5}) \Rightarrow \cos\alpha = \sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}$$

$$\Rightarrow \alpha = \pi/10 = 18° \qquad \Rightarrow \beta = \pi - \theta - \alpha = \pi/10 = 18°$$

ii) suppose $\vec{W} = a\vec{i} + b\vec{j} + c\vec{k}$ $\qquad a, b, c \in \mathbb{R}$

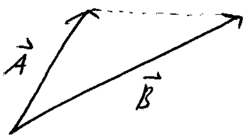
13

$$\left.\begin{aligned} |\vec{w}| &= 1 = a^2 + b^2 + c^2 \\ \vec{w}\cdot\vec{u} &= 0 = aT_+ - b + cT_- \\ \vec{w}\cdot\vec{v} &= -\tfrac{1}{2} = (-aT_+ + b + cT_-)/2 \end{aligned}\right\} \Rightarrow \vec{w} = \tfrac{1}{2}\left(1, T_-, -T_+\right)$$

or $\vec{w} = \frac{1}{2}\left(\frac{2}{\sqrt{5}} - 1, -\frac{1}{2} - \frac{3}{2\sqrt{5}}, -T_+\right)$

4)

5



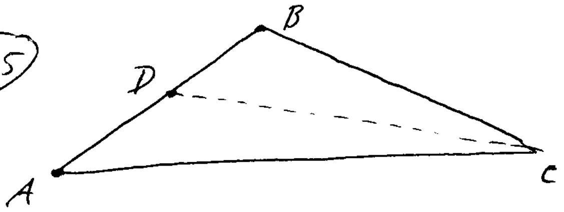
$$\vec{AB} = \vec{B} - \vec{A} = 2\vec{i} - 2\vec{j} + (m-3)\vec{k}$$

$$\vec{F} = 5\vec{i} - 3\vec{j} + 2\vec{k}$$

$$\vec{F}\cdot\vec{AB} = 10 + 6 + 2m - 6 = 30 \Rightarrow m = 10$$

5)

25



$$|\vec{AC}|^2 = (\vec{AD} + \vec{DC})\cdot(\vec{AD} + \vec{DC})$$
$$= |\vec{AD}|^2 + |\vec{DC}|^2 + 2\,\vec{AD}\cdot\vec{DC}$$
$$|\vec{BC}|^2 = |\vec{BD}|^2 + |\vec{DC}|^2 + 2\,\vec{BD}\cdot\vec{DC}$$

$$\Rightarrow |\vec{AC}|^2|\vec{BD}| = |\vec{AD}|^2|\vec{BD}| + |\vec{DC}|^2|\vec{BD}| + 2\,\vec{AD}\cdot\vec{DC}\,|\vec{BD}| \qquad (1)$$

$$|\vec{BC}|^2|\vec{AD}| = |\vec{BD}|^2|\vec{AD}| + |\vec{DC}|^2|\vec{AD}| + 2\,\vec{BD}\cdot\vec{DC}\,|\vec{AD}| \qquad (2)$$

$$(1)+(2):\ |\vec{AC}|^2|\vec{BD}| + |\vec{BC}|^2|\vec{AD}| = |\vec{AD}|^2|\vec{BD}| + |\vec{BD}|^2|\vec{AD}| + |\vec{DC}|^2\underbrace{(|\vec{AD}| + |\vec{BD}|)}_{|\vec{AB}|}$$

$$+\underbrace{2\,\vec{AD}\cdot\vec{DC}\,|\vec{BD}| + 2\,\vec{BD}\cdot\vec{DC}\,|\vec{AD}|}_{0\ \because\ \frac{\vec{DB}}{|\vec{DB}|} + \frac{\vec{DA}}{|\vec{DA}|} = 0}$$

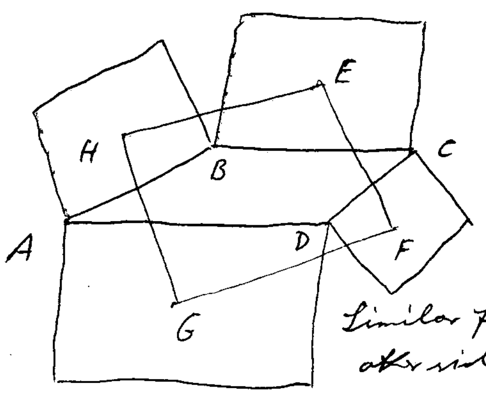
$$\Leftrightarrow |\vec{DA}|\,\vec{DB} + \vec{DA}\,|\vec{DB}| = 0$$

$$\Rightarrow |\vec{AC}|^2|\vec{BD}| + |\vec{BC}|^2|\vec{AD}| - |\vec{DC}|^2|\vec{AB}| = |\vec{AD}|^2|\vec{BD}| + |\vec{BD}|^2|\vec{AD}|$$
$$= |\vec{AD}|\,|\vec{BD}|\,\underbrace{(|\vec{AD}| + |\vec{BD}|)}_{|\vec{AB}|}$$

q.e.d

6)

25



Similar for other sides

$|\vec{EF}| = |\vec{EH}|$ $\Leftarrow$

$$\vec{EF} = -\tfrac{1}{2}(\vec{AD})^{\perp} + \tfrac{1}{2}\vec{AD} - \tfrac{1}{2}\vec{AB} - \tfrac{1}{2}(\vec{AB})^{\perp}$$

$$\vec{EH} = -\tfrac{1}{2}(\vec{AD})^{\perp} - \tfrac{1}{2}\vec{AD} - \tfrac{1}{2}\vec{AB} + \tfrac{1}{2}(\vec{AB})^{\perp}$$

$$\left.\begin{aligned} |\vec{EF} + \vec{EH}| &= |(\vec{AD})^{\perp} + \vec{AB}| \\ |\vec{EF} - \vec{EH}| &= |\vec{AD} - \vec{AB}^{\perp}| = |(\vec{AD} - \vec{AB}^{\perp})^{\perp}| \end{aligned}\right\} =$$