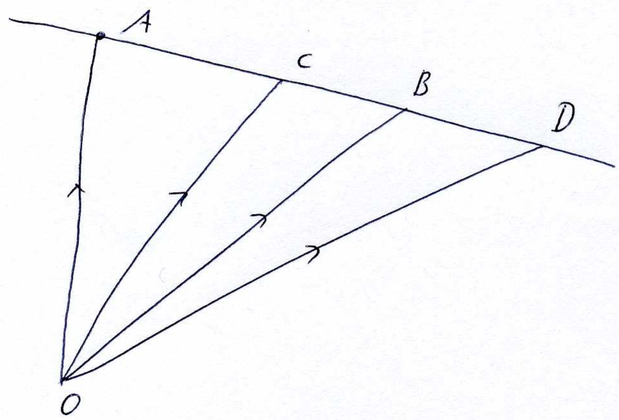
G & V coursework 1 solutions

1) i)



$\vec{OA} = 6\vec{i} + 4\vec{j}$

$\vec{OB} = 12\vec{i} + 16\vec{j}$

$AC : CB = 2:1 \Rightarrow \vec{AC} = 2\vec{CB}$

$\vec{AC} = \vec{OC} - \vec{OA} = 2\vec{CB} = 2\vec{OB} - 2\vec{OC}$

$\Rightarrow 3\vec{OC} = 2\vec{OB} + \vec{OA} = 24\vec{i} + 32\vec{j} + 6\vec{i} + 4\vec{j}$

$\Rightarrow \vec{OC} = 10\vec{i} + 12\vec{j}$ 7

ii) $\vec{AC} = \frac{\lambda}{\mu}\vec{CB}$ $\quad \vec{AD} = -\frac{\lambda}{\mu}\vec{DB} = \frac{\lambda}{\mu}\vec{BD}$

$\vec{AC} = -\vec{OA} + \vec{OC} = -6\vec{i} - 4\vec{j} + 10\vec{i} + 12\vec{j} = 4\vec{i} + 8\vec{j}$

$\vec{CB} = -\vec{OC} + \vec{OB} = -10\vec{i} - 12\vec{j} + 12\vec{i} + 16\vec{j} = 2\vec{i} + 4\vec{j}$

with $\vec{AC} = \frac{\lambda}{\mu}\vec{CB} \iff 4\vec{i} + 8\vec{j} = \frac{\lambda}{\mu}(2\vec{i} + 4\vec{j}) \Rightarrow \frac{\lambda}{\mu} = 2$

$\vec{AD} = \vec{AB} + \vec{BD} = \vec{AB} + \frac{\mu}{\lambda}\vec{AD} \Rightarrow \vec{AD} = 2\vec{AB} = -\vec{OA} + \vec{OD}$

$\Rightarrow \vec{OD} = 2\vec{AB} + \vec{OA} = -2\vec{OA} + 2\vec{OB} + \vec{OA}$

$= 2\vec{OB} - \vec{OA} = 24\vec{i} + 32\vec{j} - 6\vec{i} - 4\vec{j}$

$\Rightarrow \vec{OD} = 18\vec{i} + 28\vec{j}$ 13

$\boxed{\Sigma = 20}$

2) i) $\sigma(\sigma(\vec{v})) = \sigma(\vec{v} - 2(\vec{v}\cdot\vec{e})\vec{e})$

$= \vec{v} - 2(\vec{v}\cdot\vec{e})\vec{e} - 2\left[(\vec{v} - 2(\vec{v}\cdot\vec{e})\vec{e})\cdot\vec{e}\right]\vec{e}$

$= \vec{v} - 2(\vec{v}\cdot\vec{e})\vec{e} - 2\left[\vec{v}\cdot\vec{e} - 2(\vec{v}\cdot\vec{e})\underbrace{\vec{e}\cdot\vec{e}}_{1}\right]\vec{e}$

$= \vec{v}$ 2

ii) $\left.\begin{aligned} \sigma(\vec{v})\cdot\vec{u} &= \vec{v}\cdot\vec{u} - 2(\vec{v}\cdot\vec{e})(\vec{e}\cdot\vec{u}) \\ \vec{v}\cdot\sigma(\vec{u}) &= \vec{v}\cdot\vec{u} - 2(\vec{u}\cdot\vec{e})(\vec{v}\cdot\vec{e}) \end{aligned}\right\} \Rightarrow \sigma(\vec{v})\cdot\vec{u} = \vec{v}\cdot\sigma(\vec{u})$ 2

iii) $|\sigma(\vec{v})|^2 = \sigma(\vec{v}) \cdot \sigma(\vec{v}) = (\vec{v} - 2(\vec{v}\cdot\vec{e})\vec{e}) \cdot (\vec{v} - 2(v\cdot\vec{e})\vec{e})$

$= \vec{v}\cdot\vec{v} - 4(\vec{v}\cdot\vec{e})(\vec{e}\cdot\vec{v}) + 4(\vec{v}\cdot\vec{e})(\vec{v}\cdot\vec{e})\underbrace{\vec{e}\cdot\vec{e}}_{1}$

$= \vec{v}\cdot\vec{v} = |\vec{v}|^2 \quad \Rightarrow \quad |\sigma(\vec{v})| = |\vec{v}|$ (2)

iv) $\sigma(\vec{v}_1) = \sigma(5\vec{i} + 4\vec{j}) = 5\vec{i} + 4\vec{j} - \underbrace{2(5\vec{i} + 4\vec{j}) \cdot \tfrac{1}{\sqrt{2}}(\vec{i} + \vec{j})}_{9} \tfrac{1}{\sqrt{2}}(\vec{i} + \vec{j})$

$= -4\vec{i} - 5\vec{j}$

$\sigma(\vec{v}_2) = \sigma(2\vec{i}) = 2\vec{i} - \underbrace{2(2\vec{i} \cdot \tfrac{1}{\sqrt{2}}(\vec{i} + \vec{j}))}_{2} \tfrac{1}{\sqrt{2}}(\vec{i} + \vec{j}) = -2\vec{j}$

$\sigma(\vec{v}_3) = \vec{i} + \vec{j} + \vec{k} - \underbrace{[2(\vec{i} + \vec{j} + \vec{k}) \cdot \tfrac{1}{\sqrt{2}}(\vec{i} + \vec{j})]}_{2} \tfrac{1}{\sqrt{2}}(\vec{i} + \vec{j}) = -\vec{i} - \vec{j} + \vec{k}$

(2)

i) means when we apply σ twice we come back to the same point

iii) means that σ preserves the length

$\Rightarrow$ σ is a reflection at the plane through the origin perpendicular to $\vec{e}$. This is also supported by the examples in iv) (2)

$\boxed{\Sigma = 10}$

3) Compute all possible vectors between two points:

$\vec{AB} = \vec{OB} - \vec{OA} = \vec{i} - \vec{j} + \vec{k} - 2\vec{i} - 3\vec{j} - 5\vec{k} = -\vec{i} - 4\vec{j} - 4\vec{k}$

$\vec{AC} = \vec{OC} - \vec{OA} = \vec{i} + \vec{j} - 2\vec{k} - 2\vec{i} - 3\vec{j} - 5\vec{k} = -\vec{i} - 2\vec{j} - 7\vec{k}$

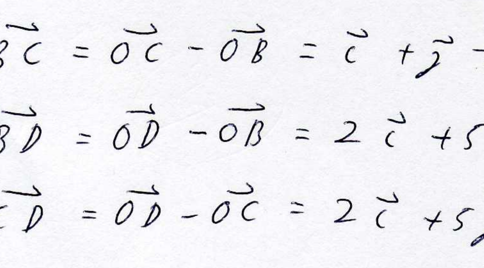
$\vec{AD} = \vec{OD} - \vec{OA} = 2\vec{i} + 5\vec{j} + 2\vec{k} - 2\vec{i} - 3\vec{j} - 5\vec{k} = 2\vec{j} - 3\vec{k}$

$\vec{BC} = \vec{OC} - \vec{OB} = \vec{i} + \vec{j} - 2\vec{k} - \vec{i} + \vec{j} - \vec{k} = 2\vec{j} - 3\vec{k}$

$\vec{BD} = \vec{OD} - \vec{OB} = 2\vec{i} + 5\vec{j} - 2\vec{k} - \vec{i} + \vec{j} - \vec{k} = \vec{i} + 6\vec{j} + \vec{k}$

$\vec{CD} = \vec{OD} - \vec{OC} = 2\vec{i} + 5\vec{j} + 2\vec{k} - \vec{i} - \vec{j} + 2\vec{k} = \vec{i} + 4\vec{j} + 4\vec{k}$

$\vec{AD} = \vec{BC} \Rightarrow \overleftrightarrow{AD} \parallel \overleftrightarrow{BC} \qquad \vec{AB} = -\vec{CD} \Rightarrow \overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$



$\Rightarrow$ no more pairs can be parallel

The plane which contains the parallelogram contains ABCD

$\Rightarrow$ The points are coplanar.

$\boxed{\Sigma = 10}$

4) We have $\vec{v} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ for $\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$

$\vec{B} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$

EXP 1:

$$q(-4\vec{j} + 2\vec{k}) = q \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ b_1 & b_2 & b_3 \end{vmatrix} = q(-\vec{j} b_3 + \vec{k} b_2)$$

$\Rightarrow$ $\underline{b_2 = 2}$, $\underline{b_3 = 4}$

EXP 2:

$$q(4\vec{i} - \vec{k}) = q \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ b_1 & 2 & 4 \end{vmatrix} = q(4\vec{i} - b_1 \vec{k}) \Rightarrow \underline{b_1 = 1}$$

$$\Rightarrow \underline{\underline{\vec{B} = \vec{i} + 2\vec{j} + 4\vec{k}}}$$

Experiment 3 should just confirm this; indeed

EXP 3:

$$q(-2\vec{i} + \vec{j}) = q \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ 1 & 2 & 4 \end{vmatrix} = q(-2\vec{i} + \vec{j}) \quad \checkmark$$

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Suppose there is a direction such that

$$q \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{F} = q(f_1 \vec{i} + f_2 \vec{j} + f_3 \vec{k})$$

$$\Rightarrow \vec{i}(v_2 b_3 - b_2 v_3) + \vec{j}(b_1 v_3 - v_1 b_3) + \vec{k}(v_1 b_2 - v_2 b_1) = f_1 \vec{i} + f_2 \vec{j} + f_3 \vec{k}$$

$$\Rightarrow \quad \begin{aligned} v_2 b_3 - b_2 v_3 &= f_1 \\ b_1 v_3 - v_1 b_3 &= f_2 \\ v_1 b_2 - v_2 b_1 &= f_3 \end{aligned}$$

For a non-trivial solution for v_1, v_2, v_3 we need to compute

$$\begin{vmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{vmatrix} = 0$$

$\boxed{\Sigma = 10}$

$\Rightarrow$ No solution exists

$\Rightarrow$ One needs at least two experiments 2