Geometry and Vectors

Coursework 1

SOLUTIONS:

1. Given are the vectors $\boxed{\sum = 12}$

$$\vec{u} = \lambda\vec{\imath} - 7\vec{\jmath} - \vec{k}, \qquad \text{and} \qquad \vec{v} = 2\vec{\imath} - \vec{\jmath} + 2\vec{k}.$$

(i) In general we have

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|}.$$

We compute

$$\left.\begin{array}{r} \vec{u}\cdot\vec{v} = 2\lambda + 7 - 2 \\ |\vec{u}| = \sqrt{\lambda^2 + 49 + 1} \\ |\vec{v}| = \sqrt{4+1+4} \end{array}\right\} \Rightarrow \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{2\lambda+5}{3\sqrt{\lambda^2+50}}.$$

Therefore

$$\frac{9}{2} = \frac{(2\lambda+5)^2}{\lambda^2+50} \Rightarrow \frac{9}{2}\lambda^2 + \frac{9}{2}50 = 4\lambda^2 + 20\lambda + 25 \Rightarrow \boxed{\lambda = 20}.$$

$\boxed{5}$

(ii) Take the unknown vector to be of the general form

$$\vec{w} = a\vec{\imath} + b\vec{\jmath} + c\vec{k} \qquad \text{with } a, b, c \in \mathbb{R}.$$

Since $\vec{u} \perp \vec{w}$ and $\vec{v} \perp \vec{w}$ we have

$$\left.\begin{array}{r} \vec{u}\cdot\vec{w} = -a - 7b - c = 0 \\ \vec{v}\cdot\vec{w} = 2a - b + 2c = 0 \end{array}\right\} \Rightarrow b = 0, a = -c.$$

The vector $\vec{w}$ has length $\sqrt{90}$

$$\vec{w}\cdot\vec{w} = 90 = a^2 + b^2 + c^2 \Rightarrow 90 = a^2 + a^2 \Rightarrow a = \pm 3\sqrt{5}.$$

Therefore

$$\boxed{\vec{w} = \pm 3\sqrt{5}(\vec{\imath} - \vec{k})}.$$

$\boxed{5}$

(*iii*) We compute

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 14 & -7 & -1 \\ 2 & -1 & 2 \end{vmatrix} = \begin{vmatrix} \vec{\imath} + 2\vec{\jmath} & \vec{\jmath} & \vec{k} \\ 0 & -7 & -1 \\ 0 & -1 & 2 \end{vmatrix}$$
$$= (\vec{\imath} + 2\vec{\jmath})(-14 - 1) = \boxed{-15(\vec{\imath} + 2\vec{\jmath})}.$$

$\boxed{2}$

$\boxed{\sum = 12}$

2. (*i*) We scalar multiply the original equation by $\vec{b}$

$$\lambda \vec{x} + (\vec{x} \cdot \vec{b})\vec{a} = \vec{c} \qquad | \cdot \vec{b} \tag{1}$$

$$\Rightarrow \lambda \vec{x} \cdot \vec{b} + (\vec{x} \cdot \vec{b})\vec{a} \cdot \vec{b} = \vec{c} \cdot \vec{b} \tag{2}$$
$$\Rightarrow \vec{x} \cdot \vec{b} = \frac{\vec{c} \cdot \vec{b}}{\lambda + \vec{a} \cdot \vec{b}} \qquad \text{for } \lambda + \vec{a} \cdot \vec{b} \neq 0$$

Substituting this into (1) gives

$$\lambda \vec{x} + \frac{\vec{c} \cdot \vec{b}}{\lambda + \vec{a} \cdot \vec{b}} \vec{a} = \vec{c} \Rightarrow \boxed{\vec{x} = \frac{1}{\lambda}\left(\vec{c} - \frac{\vec{c} \cdot \vec{b}}{\lambda + \vec{a} \cdot \vec{b}}\right) \qquad \text{for } \lambda + \vec{a} \cdot \vec{b} \neq 0}.$$

$\boxed{6}$

When $\lambda + \vec{a} \cdot \vec{b} = 0$ it follows from (2) that $\vec{c} \cdot \vec{b} = 0$

$$\Rightarrow \boxed{\vec{x} = \frac{1}{\lambda}\vec{c} + \kappa \vec{a} \qquad \text{for } \kappa \in \mathbb{R},\ \lambda + \vec{a} \cdot \vec{b} = 0}.$$

$\boxed{2}$

(*ii*) We cross multiply the original equation by $\vec{a}$ from the left

$$\vec{a} \times \vec{x} \times \vec{a} = \vec{a} \times \vec{b}. \tag{3}$$

Using the general identity

$$\vec{u} \times \vec{v} \times \vec{w} = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$$

we can re-write (3) as

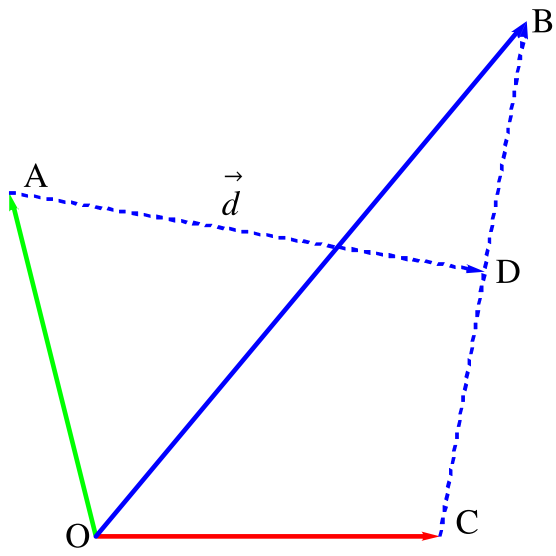
$$(\vec{a} \cdot \vec{a})\vec{x} - (\vec{a} \cdot \vec{x})\vec{a} = \vec{a} \times \vec{b}.$$

Comparing with (1), we identify $\lambda = \vec{a} \cdot \vec{a}$ and $\vec{b} = -\vec{a}$, such that $\lambda + \vec{a} \cdot \vec{b} = 0$. The solution is therefore

$$\boxed{\vec{x} = \frac{1}{\vec{a} \cdot \vec{a}}\vec{a} \times \vec{b} + \kappa \vec{a} \qquad \text{for } \kappa \in \mathbb{R}}.$$

$\boxed{4}$

3. (*i*) $\boxed{\sum = 26}$



$\boxed{3}$

(*ii*) In general we have

$$|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}| \sin\theta.$$

For $\theta = \pi/2$ we can use this to compute

$$\left|\vec{d} \times \overrightarrow{CD}\right| = \left|\vec{d}\right| \left|\overrightarrow{CD}\right|$$

From figure $\overrightarrow{CD} = \lambda\overrightarrow{CB} = \lambda(\vec{b} - \vec{c})$ for some $\lambda \in \mathbb{R}$. Therefore

$$\left|\vec{d}\right| = \frac{\left|\vec{d} \times \lambda(\vec{b} - \vec{c})\right|}{\left|\lambda(\vec{b} - \vec{c})\right|} = \frac{\left|\vec{d} \times (\vec{b} - \vec{c})\right|}{\left|\vec{b} - \vec{c}\right|}. \tag{4}$$

We also read off the figure

$$\vec{d} = -\vec{a} + \vec{c} + \lambda(\vec{b} - \vec{c}) \tag{5}$$

and compute

$$\begin{aligned} \vec{d} \times \vec{c} &= -\vec{a} \times \vec{c} + \vec{c} \times \vec{c} + \lambda(\vec{b} \times \vec{c} - \vec{c} \times \vec{c}) \\ \vec{d} \times \vec{b} &= -\vec{a} \times \vec{b} + \vec{c} \times \vec{b} + \lambda(\vec{b} \times \vec{b} - \vec{c} \times \vec{b}). \end{aligned}$$

With $\vec{c} \times \vec{c} = \vec{b} \times \vec{b} = 0$ we obtain

$$\begin{aligned} \vec{d} \times (\vec{b} - \vec{c}) &= -\vec{a} \times \vec{b} + \vec{c} \times \vec{b} - \lambda\vec{c} \times \vec{b} + \vec{a} \times \vec{c} - \lambda\vec{b} \times \vec{c} \\ &= -\left(\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right). \end{aligned}$$

Therefore with (4) follows

$$\boxed{\left|\vec{d}\right| = \frac{\left|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right|}{\left|\vec{b}-\vec{c}\right|}}.$$

$\boxed{10}$

(iii) Compute

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ -\frac{1}{4} & 1 & 0\\ 1 & 0 & 0\end{vmatrix}=-\vec{k},\vec{b}\times\vec{c}=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ 1 & 0 & 0\\ \frac{5}{4} & \frac{3}{2} & 0\end{vmatrix}=\frac{3}{2}\vec{k},\vec{c}\times\vec{a}=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ \frac{5}{4} & \frac{3}{2} & 0\\ -\frac{1}{4} & 1 & 0\end{vmatrix}=\frac{13}{8}\vec{k}.$$

Therefore

$$\left.\begin{array}{l}\left|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right|=\frac{17}{8}\\ \left|\vec{b}-\vec{c}\right|=\left|-\frac{1}{4}\vec{\imath}-\frac{3}{2}\vec{\jmath}\right|=\frac{\sqrt{37}}{4}\end{array}\right\}\Rightarrow\boxed{\left|\vec{d}\right|=\frac{17}{2\sqrt{37}}}.$$

5

(iv) From (5)

$$\begin{aligned}\vec{d}&=-\vec{a}+\vec{c}+\lambda(\vec{b}-\vec{c})\\ &=\frac{1}{4}\vec{\imath}-\vec{\jmath}+\frac{5}{4}\vec{\imath}+\frac{3}{2}\vec{\jmath}+\lambda\left(-\frac{1}{4}\vec{\imath}-\frac{3}{2}\vec{\jmath}\right)=\left(\frac{3}{2}-\frac{1}{4}\lambda\right)\vec{\imath}+\left(\frac{1}{2}-\frac{3}{2}\lambda\right)\vec{\jmath}\end{aligned}$$

Then

$$\begin{aligned}&\Rightarrow\vec{d}\cdot\vec{d}=\left(\frac{3}{2}-\frac{1}{4}\lambda\right)^2+\left(\frac{1}{2}-\frac{3}{2}\lambda\right)^2=\frac{17^2}{4\cdot 37}\\ &\Rightarrow\frac{17^2}{4\cdot 37}=\frac{5}{2}-\frac{9}{4}\lambda+\frac{37}{16}\lambda^2\Rightarrow\lambda=\frac{18}{37}\\ &\Rightarrow\overrightarrow{OD}=\vec{d}+\vec{a}=\left(\frac{3}{2}+\frac{1}{4}\frac{18}{37}\right)\vec{\imath}+\left(\frac{1}{2}-\frac{3}{2}\frac{18}{37}\right)\vec{\jmath}-\frac{1}{4}\vec{\imath}+\vec{\jmath}\\ &\Rightarrow\boxed{\overrightarrow{OD}=\frac{167}{148}\vec{\imath}+\frac{57}{74}\vec{\jmath}}.\end{aligned}$$

8