Generalised permutation branes

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To appear soon...

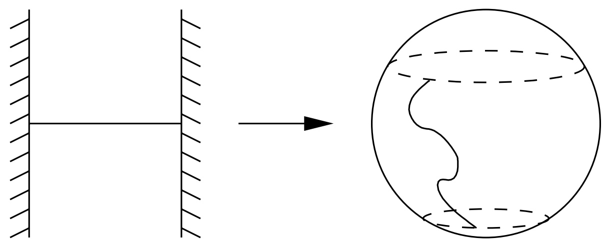
The issue of classifying branes

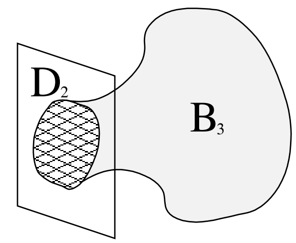
- Major steps in the classification of branes:
  - Untwisted branes (Cardy's construction)
  - Twisted branes
  - [(Super)conformal branes (only for very special models)]
  - Symmetry breaking branes
  - What else?

  $\Rightarrow$ Need to look for branes with special properties
- **Proposal:** Employ predictions from K-theory
  - K-groups for group manifolds are known explicitly
  - Charge carrying branes should have an enhanced symmetry
  - One can get some hints about the geometry
  - **Prototype:** $SU(2) \times SU(2)$ WZW model
    $\Rightarrow$ Full control for equal levels, but otherwise?

A short introduction to WZW models

- WZW models: 2d non-linear σ-models on a Lie group G





- Action: Based on non-degenerate invariant form $\langle \cdot, \cdot \rangle$,

$$\mathcal{S}[g] \sim \int_{\Sigma_2} \mathcal{L}_{\text{kin}} + \int_{B_3} \omega^{\text{WZ}} - \int_{D_2} \omega_2$$

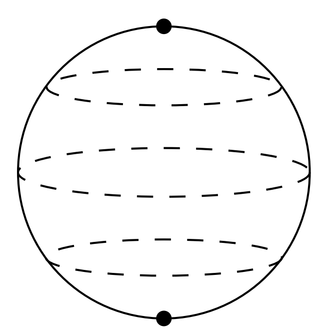
- WZ-form: $3\omega^{\text{WZ}} = \langle g^{-1}dg, [g^{-1}dg, g^{-1}dg] \rangle$, $\qquad d\omega_2 = \omega^{\text{WZ}}\big|_{D_2}$
- For G simple: $\langle \cdot, \cdot \rangle \ = \ k\, \kappa(\cdot, \cdot)$
- $\hat{G}_k \times \hat{G}_k$ loop group symmetry $g \mapsto g_L(z) g g_R(\bar{z})^{-1}$

Maximally symmetric branes on group manifolds

- Branes wrap twisted conjugacy classes

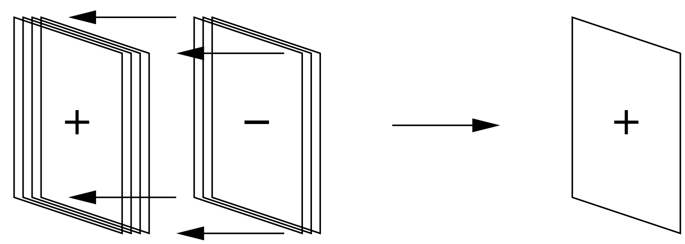
$$\mathcal{D} \;=\; \mathcal{C}_f(\Omega) \;=\; \left\{ hf\Omega(h^{-1}) \middle| h \in G \right\}$$

- Diagonal $\hat{G}_k$ loop group symmetry via $g \mapsto hg\Omega(h^{-1})$
- Ω must be *isometric*
- f (and hence ω_2) is quantised
- An example: $SU(2) \cong S^3$
  - Branes are labelled by integers $\lambda = 0, \ldots, k$
  - $\lambda = 0, k$ correspond to 0-branes, the rest to S^2-branes

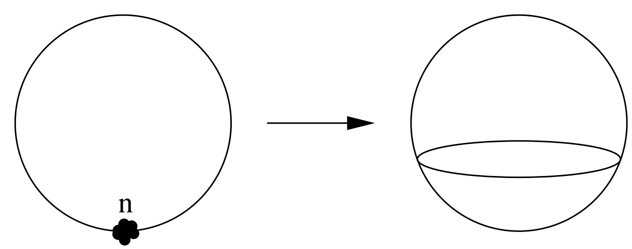


Boundary RG flow invariants and K-theory

- Basic ideas:
  - Space-time physics (tachyon condensation) $\leftrightarrow$ RG flows
  - Conserved charges $\leftrightarrow$ K-theory: $nV \oplus m\bar{V} \cong (n-m)V$



  - Non-trivial H-flux $\Rightarrow$ need to use *twisted* K-theory
- An example: $SU(2)$
  - Every brane can be obtained from 0-branes



  - Charge group $K^\tau\left(SU(2)_k\right) = \mathbb{Z}_{k+2}$

Product groups: Charge groups from K-theory

- For a simple group G one has

$$K^\tau(G_k) \;=\; (\mathbb{Z}_d)^{2^{r-1}}$$

where $r =$ rank G and d is determined by G and k

- A generalisation of the Künneth formula implies

$$K^\tau(G_{k_1} \times G_{k_2}) \;=\; \left(\mathbb{Z}_{\text{gcd}(d_1,d_2)}\right)^{2^{2r-1}}$$

- For $SU(2) \times SU(2)$ one thus obtains

$$K^\tau\big(SU(2)_{k_1} \times SU(2)_{k_2}\big) \;=\; 2 \cdot \mathbb{Z}_{\text{gcd}(k_1+2,k_2+2)}$$

- **Question:** Which branes correspond to the “new” charges?

Product groups: The case of equal levels

- Consider the product group $G \times G$ with metric

$$\langle \cdot , \cdot \rangle \;=\; k\big(\kappa_1(\cdot,\cdot) + \kappa_2(\cdot,\cdot)\big)$$

- What kind of automorphisms do we have?
  - $\Omega = \Omega_1 \times \Omega_2 \quad \Rightarrow \quad$ factorising branes

$$\mathcal{D} \;=\; \mathcal{C}_{f_1}(\Omega_1) \times \mathcal{C}_{f_2}(\Omega_2)$$

  - Exchange automorphism $\Omega(g_1, g_2) = (g_2, g_1)$

$$\mathcal{D} \;=\; \mathcal{C}_f(\Omega) \;=\; \big\{(g_1 f g_2^{-1}, g_2 f g_1^{-1})\big| g_1, g_2 \in G\big\}$$

  - The simplest permutation brane is given by $f = 1$,

$$\mathcal{D} \;=\; \big\{(g, g^{-1})\big| g \in G\big\}$$

Product groups: The case of different levels

- Now consider the same group $G \times G$ with metric

$$\langle \cdot, \cdot \rangle \; = \; k_1 \, \kappa_1(\cdot, \cdot) + k_2 \, \kappa_2(\cdot, \cdot)$$

- What kind of automorphisms do we have now?
  - $\Omega = \Omega_1 \times \Omega_2 \quad \Rightarrow \quad$ factorising branes

$$\mathcal{D} \; = \; \mathcal{C}_{f_1}(\Omega_1) \times \mathcal{C}_{f_2}(\Omega_2)$$

  - The exchange automorphism $\Omega(g_1, g_2) = (g_2, g_1)$ still exists but it is *not isometric* anymore!
  - **Proposal:** The simplest permutation brane is deformed to

$$\boxed{\mathcal{D} \; = \; \left\{ (g^{k_2'}, g^{-k_1'}) \middle| g \in G \right\} \; \text{with} \; k_i' = k_i / \text{gcd}(k_1, k_2)}$$

  - The symmetry is broken to $\hat{G}_{k_1+k_2}$

Lagrangian approach

- Trivialising two-form for the simplest permutation brane

$$\omega_2 \;=\; k_1 \sum_{j=1}^{k_2'-1} (k_2' - j)\,\mathrm{tr}\big[\mathrm{Ad}_{g^j}(g^{-1}dg)g^{-1}dg\big] + (1 \leftrightarrow 2)$$

- One can prove gluing conditions $J_1 + J_2 = \bar{J}_1 + \bar{J}_2$
- Concrete formulas for $SU(2) \times SU(2)$:
  - $g = \begin{pmatrix} \cos\psi + i\cos\theta\sin\psi & \sin\psi\sin\theta e^{i\phi} \\ -\sin\psi\sin\theta e^{-i\phi} & \cos\psi - i\cos\theta\sin\psi \end{pmatrix} \Rightarrow g(\psi)^n = g(n\psi)$
  - $ds^2 = \sum k_i\big(d\psi_i^2 + \sin^2\psi(d\theta_i^2 + \sin^2\theta_i d\phi_i)\big)$
  - $B = \sum k_i\big(\psi_i - \frac{1}{2}\sin(2\psi_i)\big)\sin\theta_i\, d\theta_i \wedge d\phi_i, \quad H = dB$
  - Brane embedding: $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} k_2'\psi \\ -k_1'\psi \end{pmatrix}$, $\theta_i = \theta$, $\phi_i = \phi$
  - $\omega_2 \sim \big[k_2\sin(2k_1'\psi) - k_1\sin(2k_2'\psi)\big]\sin\theta\, d\theta \wedge d\phi$

Dirac-Born-Infeld approach

- Branes minimise the DBI action

$$\mathcal{S}_{\mathcal{D}} = \int e^{-\Phi}\sqrt{\det(\hat{g}+\omega_2)} + \cdots$$

  - Brane embedding: $X^\mu = X^\mu(Y^i)$
  - Induced metric $\hat{g}_{ij} = g_{\mu\nu}\partial_i X^\mu \partial_j X^\nu$

- For $\Phi = 0$ the EOM read

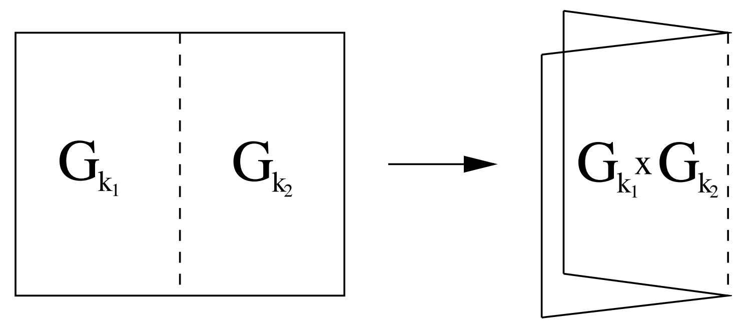
$$\mathrm{tr}\Big[(\hat{g}+\omega_2)^{-1}\Omega^\mu\Big] = 0$$

  - $\Omega^\mu_{ij} = \partial_i\partial_j X^\mu + \Gamma^\mu_{\nu\rho}\partial_i X^\nu \partial_j X^\rho - \hat{\Gamma}^k_{ij}\partial_k X^\mu$
  - Generalised connection $\Gamma = \Gamma_{LC}(g) - \frac{1}{2}H$

- Works out for the simplest permutation brane on $G \times G$!

Non-trivial defect lines from folding

- Folding maps defect lines between CFT$_1$ and CFT$_2$ to boundaries in CFT$_1$ $\times$ CFT$_2$

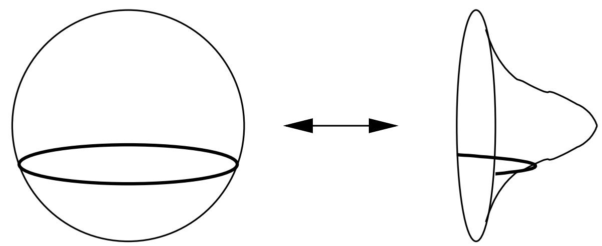


- Our construction gives rise to new non-trivial defects between two WZW models based on the same group but at different levels.
- *Reminder:* The old construction used the decomposition

$$\frac{G_{k_1} \times G_{k_2}}{G_{k_1+k_2}} \times G_{k_1+k_2} \hookrightarrow G_{k_1} \times G_{k_2}$$

Generalisation to cosets

- How to extend our construction to cosets $G/H \times G/H$?
  - The group G and the coset G/T ($T =$ maximal torus) are related by a marginal bulk deformation



  - Natural guess for the brane geometry:

$$\mathcal{D} \; = \; \Big\{ \big((gh)^{k'_2}, (hg)^{-k'_1} \big) \Big| g \in G, h \in H \Big\} \; \subset \; G \times G$$

  - *But:* Open whether DBI is satisfied for this proposal
- **Aim:** New branes in Calabi-Yau manifolds (Gepner models)

$$\frac{SU(2)_{k_1} \times U(1)_2}{U(1)_{k_1+2}} \times \cdots \times \frac{SU(2)_{k_n} \times U(1)_2}{U(1)_{k_n+2}}$$

Towards a CFT description?

- Denote by MM_p the p^{th} minimal model
- Consider the following CFT

$$\text{MM}_p \times \text{MM}_p \ \text{ with a permutation brane}$$

- Then one should be able to follow the combined bulk/boundary flow to the CFT ($q < p$)

$$\text{MM}_q \times \text{MM}_p \ \text{ with some brane configuration } X$$

- **Questions:**
  - How does X look like?
  - Is it an elementary brane or composite?
  - Can one identify the symmetry preserved?

Outlook

- Unfortunately not enough time to discuss:
  - Higher dimensional branes ($f \neq 1$)
  - Arbitrary permutations in $G \times \cdots \times G$
- Open questions:
  - DBI calculations for *all* these branes
    - Check of the EOM
    - Calculation of energies
  - CFT description
    - Identification of the precise symmetry preserved
    - Boundary states
    - Calculation of boundary entropies (g-factors)
    - Comparison with DBI results
  - Application to Gepner models
  - Analogous construction for principal chiral models?