A new viewpoint on form factors and correlation functions in the Ising field theory at finite temperature

Benjamin Doyon

Rudolf Peierls Centre for Theoretical Physics, Oxford University, UK
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Correlation functions and form factors

Euclidean correlation functions at zero temperature:

$$\langle\mathcal{O}(x,\tau)\mathcal{O}(0,0)\rangle = \sum_{k=0}^{\infty}\int \frac{d\theta_1\cdots d\theta_k}{k!}e^{-Mr\sum_j \cosh(\theta_j)}\times$$

$$\times\langle\mathrm{vac}|\mathcal{O}(0,0)|\theta_1,\ldots,\theta_k\rangle_{in\ in}\langle\theta_1,\ldots,\theta_k|\mathcal{O}(0,0)|\mathrm{vac}\rangle$$

where $r=\sqrt{x^2+\tau^2}$.

Useful representation in integrable models because:

- **form factors** $\langle\mathrm{vac}|\mathcal{O}(0,0)|\theta_1,\ldots,\theta_k\rangle_{in}$ of local fields **can be evaluated** (up to normalization) by solving a Riemann-Hilbert problem in rapidity space;
- the first few terms ($k=0,\ 1,\ 2,\ \ldots$) in the expression above give a **good description of correlation functions at large (and also not too large) distances**;
- in condensed matter applications, for instance, the **most relevant region is the large-distance one (low energy)**.

Finite (non-zero) temperature

Goal:

Find a **large-distance expansion** of finite-temperature correlation functions, and compute the terms ("form factors") involved.

- Must take statistical average as well:

$$\langle\langle \mathcal{O}(x,\tau)\mathcal{O}(0,0)\rangle\rangle_L = \frac{\mathrm{Tr}\left(e^{-LH}\mathcal{O}(x,\tau)\mathcal{O}(0,0)\right)}{\mathrm{Tr}\left(e^{-LH}\right)} .$$

  Not a vacuum-vacuum matrix element: no obvious simple form factor decomposition.

  Geometrically, the trace represents a theory on a **cylinder of circumference** L where time is around the cylinder and space is along it.

- Different quantization scheme for the same theory: take **time along the cylinder** and **space around the cylinder**,

$$\langle\langle \mathcal{O}(x,\tau)\cdots\rangle\rangle_L = {}_L\langle \mathrm{vac}|e^{i\pi s/2}\mathcal{O}_L(-\tau,x)\cdots|\mathrm{vac}\rangle_L$$

  where s is the spin of $\mathcal{O}$ and $\mathcal{O}_L$ is the operator on the cylinder associated to $\mathcal{O}$.

Correlation functions and form factors on the cylinder

Form factor decomposition on the cylinder (integrable models):

$$
{}_L\langle \mathrm{vac}|\mathcal{O}_L(x,\tau)\mathcal{O}_L(0,0)|\mathrm{vac}\rangle_L = \sum_{k=0}^{\infty}\sum_{n_1,\ldots,n_k} e^{\sum_j n_j \frac{2\pi i x}{L} - E_{n_1,\ldots,n_k}\tau} \times
$$

$$
\times \; {}_L\langle \mathrm{vac}|\mathcal{O}_L(0,0)|n_1,\ldots,n_k\rangle\langle n_1,\ldots,n_k|\mathcal{O}_L(0,0)|\mathrm{vac}\rangle_L
$$

Problems:

- **Spectrum** $E_{\{n\}}$ is **complicated** in interacting models;
- **Form factors** ${}_L\langle \mathrm{vac}|\mathcal{O}_L(0,0)|n_1,\ldots,n_k\rangle$ depend on **discrete variables**: no obvious “analytical way” of evaluating them.

What has been done

On the cylinder:

Form factors of spin fields in the Ising field theory:

- A. I. Bugrij: hep-th/0011104, hep-th/0107117.
- A. B. Zamolodchikov and P. Fonseca: hep-th/0112167.

Also, spectrum in interacting integrable models by various numerical means (TBA, NLIE, ...)

"Finite temperature form factor" approach:

- A. Leclair, F. Lesage, S. Sachdev, H. Saleur: Nucl. Phys. B482 [FS] 1996, 579.
- A. Leclair, G. Mussardo: Nucl. Phys. B552 1999, 624.
- H. Saleur: Nucl. Phys. B567 2000, 602.
- G. Delfino: J. Phys. A 34, 2001, L161.
- G. Mussardo: J. Phys. A 34 2001, 7399.
- O. A. Castro-Alvaredo and A. Fring: Nucl. Phys. B636 [FS] 2002, 611
- R. A. J. van Elburg and K. Schoutens: cond-mat/0007226.

General idea

- Find a construction where the trace
$$\frac{\mathrm{Tr}\left(e^{-LH}\mathcal{O}(x,\tau)\mathcal{O}(0,0)\right)}{\mathrm{Tr}\left(e^{-LH}\right)}\,.$$
is a **vacuum expectation value** in some vector space and where **eigenvalues of the momentum operator** are described by **continuous variables** (for instance, by **rapidities** θ_j).

- [Ising field theory] Find a measure $\rho(\{\theta\})$ in
$$\mathbf{1} = \int \frac{\{d\theta\}}{n!}\rho(\{\theta\})|\{\theta\}\rangle\langle\{\theta\}|$$
such that form factors of uninteracting local fields, $\langle\mathrm{vac}|\mathcal{O}(0,0)|\{\theta\}\rangle$, are **entire functions** of the rapidities $\{\theta\}$;

- Calculate form factors on the cylinder by **analytical continuation** in the rapidity variables to the positions of the **poles of the measure** ρ:

$$_L\langle\mathrm{vac}|\mathcal{O}_L(0,0)|\{n\}\rangle \propto \sqrt{\mathrm{Res}\,\rho}\,\langle\mathrm{vac}|\mathcal{O}(0,0)|\{\alpha_n \pm i\pi/2\}\rangle$$

Ising field theory

Free Majorana fermion operators on the line:

$$\psi(x,\tau) = \frac{1}{2}\sqrt{\frac{m}{\pi}} \int d\theta\, e^{\theta/2} \left(a(\theta)\, e^{ip_\theta x - E_\theta \tau} + a^\dagger(\theta)\, e^{-ip_\theta x + E_\theta \tau} \right)$$

$$\bar{\psi}(x,\tau) = -\frac{i}{2}\sqrt{\frac{m}{\pi}} \int d\theta\, e^{-\theta/2} \left(a(\theta)\, e^{ip_\theta x - E_\theta \tau} - a^\dagger(\theta)\, e^{-ip_\theta x + E_\theta \tau} \right)$$

$$\{a^\dagger(\theta), a(\theta')\} = \delta(\theta - \theta')\ ,$$

$$p_\theta = m \sinh\theta\ , \quad E_\theta = m\cosh\theta\ .$$

Satisfy equations of motion and
equal-time canonical anti-commutation relations

$$\begin{aligned} \bar{\partial}\psi(x,\tau) &\equiv \frac{1}{2}\left(\partial_x + i\,\partial_\tau\right)\psi = \frac{m}{2}\bar{\psi} \\ \partial\bar{\psi}(x,\tau) &\equiv \frac{1}{2}\left(\partial_x - i\,\partial_\tau\right)\bar{\psi} = \frac{m}{2}\psi \end{aligned}$$

$$\{\psi(x), \psi(x')\} = \delta(x - x')\ , \quad \{\bar{\psi}(x), \bar{\psi}(x')\} = \delta(x - x')\ .$$

Hilbert space $\mathcal{H}$: Fock space over mode algebra.

Hamiltonian:

$$H = m \int d\theta \cosh(\theta) a^\dagger(\theta) a(\theta)\ .$$

Space of operators

Consider the vector space $\mathcal{L}$ of operators of the theory:

- **Vacuum:**

$$|\mathrm{vac}_{\mathcal{L}}\rangle \;\equiv\; \mathbf{1}_{\mathcal{H}}$$

- **Complete basis:**

$$|\theta_1, \ldots, \theta_N\rangle^{\sim}_{\epsilon_1,\ldots,\epsilon_N} \;\equiv\; a^{\epsilon_1}(\theta_1)\cdots a^{\epsilon_N}(\theta_N)$$

$$\theta_1 > \theta_2 > \cdots > \theta_N$$

  where ϵ_j are signs ($\pm$: "particles / holes") and $a^{+}(\theta) = a^{\dagger}(\theta),\; a^{-}(\theta) = a(\theta)$.

- **Inner product on $\mathcal{L}$**

$$\langle u|v\rangle \equiv \langle\langle U^{\dagger} V\rangle\rangle_L \;, \quad \text{if } u \equiv U,\; v \equiv V \;.$$

- **Operators $\mathcal{O}$ on the Hilbert space $\mathcal{H}$ can be seen also as operators on $\mathcal{L}$: acting by left-action**

Two-point function is a **vacuum expectation value** on $\mathcal{L}$:

$$\langle\langle \mathcal{O}(x)\mathcal{O}(0)\rangle\rangle_L = \langle \mathrm{vac}_{\mathcal{L}}|\mathcal{O}(x)\mathcal{O}(0)|\mathrm{vac}_{\mathcal{L}}\rangle$$

Finite-temperature form factors

Normalized set of states on $\mathcal{L}$:

$$|\theta_1,\ldots,\theta_N\rangle_{\epsilon_1,\ldots,\epsilon_N} \equiv \prod_{j=1}^{N}(1+e^{-\epsilon_j L E_{\theta_j}})a^{\epsilon_1}(\theta_1)\cdots a^{\epsilon_N}(\theta_N)$$

with $|\theta_1,\ldots,\theta_N\rangle_{\epsilon_1,\ldots,\epsilon_N} = 0$ if $\theta_i = \theta_j$ for some $i \neq j$.

Finite-temperature form factors:

$$f^{\mathcal{O}}_{\epsilon_1,\ldots,\epsilon_N}(\theta_1,\ldots,\theta_N) = \langle \mathrm{vac}_{\mathcal{L}}|\mathcal{O}(0)|\theta_1,\ldots,\theta_N\rangle_{\epsilon_1,\ldots,\epsilon_N}$$

Decomposition of the identity:

Note that

$$\langle\langle a(\theta)a^\dagger(\theta')\rangle\rangle_L = \frac{\delta(\theta-\theta')}{1+e^{-LE_\theta}},\ \langle\langle a^\dagger(\theta)a(\theta')\rangle\rangle_L = \frac{\delta(\theta-\theta')}{1+e^{LE_\theta}}$$

Then

$$_\epsilon\langle\theta|\theta'\rangle_{\epsilon'} = (1+e^{-\epsilon L E_\theta})\delta(\theta-\theta')\delta_{\epsilon,\epsilon'}$$

In general, we have

$$\mathbf{1} = \sum_{\{\epsilon\}}\int \frac{\{d\theta\}}{N!}\prod_{j=1}^{N}\frac{1}{1+e^{-\epsilon_j L E_{\theta_j}}}|\{\theta\}\rangle_{\{\epsilon\}}\ {}_{\{\epsilon\}}\langle\{\theta\}|$$

Large-distance expansion?

Can write two-point function as

$$\langle\langle \mathcal{O}(x)\mathcal{O}(0)\rangle\rangle_L =$$
$$\sum_{\{\epsilon\}} \int \frac{\{d\theta\}}{N!} \prod_{j=1}^{N} \frac{e^{i\epsilon_j p_{\theta_j} x}}{1+e^{-\epsilon_j L E_{\theta_j}}} f^{\mathcal{O}}_{\{\epsilon\}}(\{\theta\})^* f^{\mathcal{O}}_{\{\epsilon\}}(\{\theta\})$$

In the limit $L \to \infty$:

- The finite-temperature form factors become **the usual form factors**:

$$\lim_{L\to\infty} f^{\mathcal{O}}_{+,\ldots,+,-,\ldots,-}(\theta_1,\ldots,\theta_{N_+},\theta_{N_++1},\ldots,\theta_N) =$$
$$\langle \theta_N,\ldots,\theta_{N_++1}|\mathcal{O}(0)|\theta_1,\ldots,\theta_{N_+}\rangle$$
$$(\theta_i \neq \theta_j \ \forall\ i \in \{1,\ldots,N_+\},\ j\in\{N_++1,\ldots,N\})\ ;$$

- The expansion above becomes **the usual form factor expansion**.

But, as in the zero-temperature case, we need to do **analytical continuation in** θ in order to get **workable large-**x **expansion**.

Form factors on the cylinder from finite-temperature form factors

Remark:

$$f^{\mathcal{O}}_{\epsilon_1,\ldots,\epsilon_N}(\theta_1,\ldots,\theta_N) = \langle\langle\{a^{\epsilon_1}(\theta_1),[a^{\epsilon_2}(\theta_2),\{\cdots,\mathcal{O}(0)\cdots\}]\}\rangle\rangle_L$$

Local uninteracting fields $\mathcal{O}_i$:

$$[\psi(x),\mathcal{O}_i(x')] = \sum_j c_i^j\ \mathcal{O}_j(x')\ \delta^{(d_i-d_j-\frac{1}{2})}(x-x')$$

Modes in terms of local fermi fields:

$$a^\pm(\theta) = \frac{1}{2}\sqrt{\frac{m}{\pi}}\ \int_{-\infty}^{\infty} dx\, e^{\pm i p_\theta x}\ (e^{\theta/2}\psi(x) \mp i e^{-\theta/2}\bar{\psi}(x))$$

$\Rightarrow$ **finite-temperature form factors of local uninteracting fields are entire functions of θ_i's.** Then (spinless fields):

$$_L\langle \text{vac}|\mathcal{O}_L(0)|n_1,\ldots,n_k\rangle = \sum_{\{\epsilon\}}\left(\frac{i\pi}{mL}\right)^{\frac{k}{2}}\prod_{j=1}^{k}\frac{1}{\sqrt{\cosh(\theta_{n_j})}}\ f^{\mathcal{O}}_{\epsilon_1,\ldots,\epsilon_k}(\alpha_{n_1}+\frac{i\pi\epsilon_1}{2},\ldots)$$

where $mL\sinh(\alpha_n) = 2\pi n,\ n\in\mathbb{Z}+\frac{1}{2}$.

Finite-temperature form factors of uninteracting fields: mixing

At zero temperature: **form factors of uninteracting fields are also entire functions of rapidities**. Are they the same as the finite-temperature form factors?

No, in general:

$$f^{\mathcal{O}}_{\{+\}}(\{\theta\}) = \langle \mathrm{vac}|(\mathcal{O}(0) + \ldots)|\{\theta\}\rangle$$

where $\ldots$ contains local fields at $x = 0$ of **lower dimension** than that of $\mathcal{O}$ and of **equal or lower spin**.

For instance: **Casimir energy** $\mathcal{E}$

$$\langle \mathrm{vac}_{\mathcal{L}}|T(0)|\mathrm{vac}_{\mathcal{L}}\rangle = \langle \mathrm{vac}|(T(0) + \mathcal{E}\mathbf{1})|\mathrm{vac}\rangle \ .$$

In general, the mixing can be described by a **mixing operator** M acting on $\mathcal{L}$:

$$|\mathcal{O}(0) + \ldots\rangle = M|\mathcal{O}(0)\rangle \quad \in \mathcal{L}$$

This is a generalization of the CFT situation, where $\mathcal{L} \simeq \mathcal{H}$ and M is written in terms of Virasoro generators and makes a transformation to the cylinder.

Spin (or twist) operators

The Majorarana theory has $\mathbb{Z}_2$ global symmetry $\psi \mapsto -\psi$, $\bar{\psi} \mapsto -\bar{\psi}$. There are two associated spin fields (twist fields): σ and μ (with non-zero form factors for even and odd particle number respectively).

On the geometry of the cylinder, every spin operator has **two realizations**: $\sigma_\pm$ and $\mu_\pm$, with branch cut on the right $(+)$ or on the left $(-)$ of the position of the operator. For instance:

$$\begin{aligned}\psi(x,\tau)\mu_+(0) &\mapsto (\psi(x,\tau)\mu(0))_+ \quad (\tau > 0)\\ \mu_+(0)\psi(x,\tau) &\mapsto -(\psi(x,\tau)\mu(0))_+ \quad (\tau < 0)\end{aligned}$$

and

$$\begin{aligned}\psi(x,\tau)\mu_+(0) &\mapsto \mathcal{C}_{\tau'=0^+\to\tau}(\psi(x,\tau')\mu(0))_+ \quad (\tau < 0)\\ \mu_+(0)\psi(x,\tau) &\mapsto -\mathcal{C}_{\tau'=0^-\to\tau}(\psi(x,\tau')\mu(0))_+ \quad (\tau > 0)\end{aligned}$$

where $\mathcal{C}$ means analytical continuation and $(\psi(x,\tau)\mu(0))_+$ is a field that defines, inside correlation functions, a function of x and τ that has a branch cut at $\tau = 0,\ x > 0$.

Finite-temperature form factors of spin operators: Riemann-Hilbert problem

Consider

$$f(\theta_1,\ldots,\theta_k) = f^{\sigma_+,\mu_+}_{\{+\}}(\theta_1,\ldots,\theta_k)\;.$$

Then,

1. $f(\ldots,\theta_i,\ldots,\theta_j,\ldots) = -f(\ldots,\theta_j,\ldots,\theta_i,\ldots)$
2. $f(\theta_1,\ldots,\theta_k)$ has poles at $\theta_j = \alpha_n + \dfrac{i\pi}{2},\ n\in\mathbb{Z}$

   and has zeroes at $\theta_j = \alpha_n + \dfrac{i\pi}{2},\ n\in\mathbb{Z}+\dfrac{1}{2}$
3. $f(\theta_1,\ldots,\theta_k+2i\pi) = -f(\theta_1,\ldots,\theta_k)$
4. $f(\theta_1,\ldots,\theta_k) \sim$

   $$\frac{(-1)^{k-1}}{\pi}\,\frac{1+e^{-LE_{\theta_{k-1}}}}{1-e^{-LE_{\theta_{k-1}}}}\,\frac{f(\theta_1,\ldots,\theta_{k-2})}{\theta_k-\theta_{k-1}-i\pi}$$
5. $f(\theta_1,\ldots,\theta_k)$ does not have poles for

   $\mathrm{Im}(\theta_j)\in[-i\pi,i\pi]$ except those mentionned above .

Also, we have in general

$$\underline{\mathrm{P}}f^{\mathcal{O}}_{\epsilon_1,\ldots,\epsilon_k}(\theta_1,\ldots,\theta_k+i\pi) = i\,\underline{\mathrm{P}}f^{\mathcal{O}}_{\epsilon_1,\ldots,-\epsilon_k}(\theta_1,\ldots,\theta_k)$$

where $\underline{\mathrm{P}}$ means principal value.

What next to do?

- Compute explicitly form factors of descendant spin fields, including mixing;
- Generalize to interacting integrable models:
  - Riemann-Hilbert problem for finite-temperature form factors?
  - gives a way of computing energy spectrum on the cylinder?
  - gives another numerically useful representation of finite-temperature correlation functions?