

SIMPLE MODULES & STABLE CATEGORIES

Suppose

$$F: D^b(B) \xrightarrow{\sim} D^b(A)$$

$$\begin{array}{ccc} \text{simples} & & \\ S_1, \dots, S_n & \longleftrightarrow & X_1, \dots, X_n \end{array}$$

Then

$$\textcircled{*} \left\{ \begin{array}{l} \text{Hom}(X_i, X_j) = \begin{cases} k & i=j \\ 0 & i \neq j \end{cases} \\ \text{Hom}(X_i, X_j[t]) = 0 \quad (t < 0) \\ \{X_i\} \text{ gen}^s D^b(A) \text{ as a} \\ \text{triang cat.} \end{array} \right.$$

$\textcircled{*}$ = "simple minded collection".

THM If A symm (i.e., $A \cong DA$ as bimodules) then any simple m. coll. comes from a derived equiv.

Example char $k = 2$

$$A = kA_4 = k[\mathbb{F}_4 \rtimes \mathbb{F}_4^*]$$

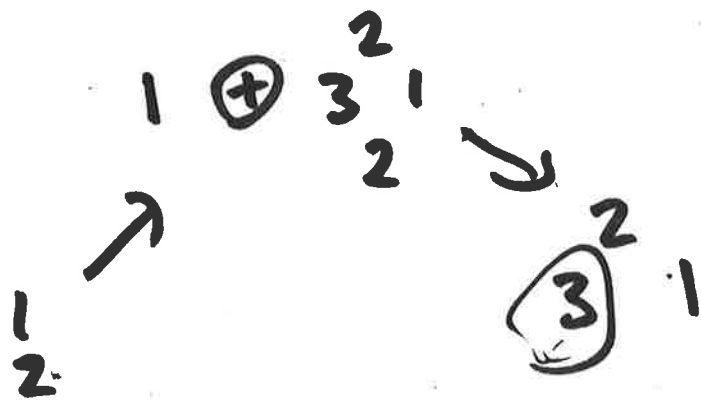
has 3 ~~one~~ one-dim simple modules

($\omega^3 = 1$) ~~1, 2, 3~~ 1, 2, 3

according to power of ω that a gen of \mathbb{F}_4^* acts by.

Proj^s $\begin{matrix} 3 \\ 1 \end{matrix} \begin{matrix} 2 \\ 2 \end{matrix}, \begin{matrix} 2 \\ 3 \end{matrix} \begin{matrix} 1 \\ 2 \end{matrix}, \begin{matrix} 1 \\ 2 \end{matrix} \begin{matrix} 3 \\ 1 \end{matrix}$

$$\chi_1 = 3, \chi_2 = \begin{matrix} 2 \\ 1 \end{matrix}, \chi_3 = \begin{matrix} 1 \\ 2 \end{matrix}$$



Suppose A, B are self-inj with no simple proj's, and \exists exact

$$F: \text{mod-}B \rightarrow \text{mod-}A$$

s.t. F induces an equiv.

$$F: \underline{\text{mod-}B} \rightarrow \underline{\text{mod-}A}$$

$$\begin{array}{c} \text{simples} \\ S_1, \dots, S_n \end{array} \leftrightarrow X_1, \dots, X_n$$

Then

$$\underline{\text{Hom}}(X_i, X_j) = \begin{cases} k & (i=j) \\ 0 & (i \neq j) \end{cases}$$

mod-}A is the smallest subcat
cont $\{X_i\}$ and closed under
extensions

(**) [i.e., for every simple A -mod S
 \exists proj P s.t. $S \oplus P$ has a filt.
with factors in $\{X_i\}$.]

(**) = "simple minded system".

Question:

Does every SMS come from a stable equiv?

If so, what can we say about A ?

Linckelmann's Lemma: If a stab eq. of Morita type sends simples to simples then it is a Morita equiv.

So if A exists, it's unique.

it with Rouquier:

Assume $\{X_i\}$ is SMS.

• For any ^{non-proj.} B -module M , there's a min. proj P_M s.t. $M \oplus P_M$ has a filt with factors in $\{X_i\}$.

• If F exists, this reconstructs $F(F^{-1}(M))$ and the filt. induced by a comp. series of $F^{-1}(M)$.

• Applying to ~~ΩX~~ ΩX ; (Example)
and sticking X_i on top, we
reconstruct a model of the Loewy
series of proj B -mods.

• This gives us the assoc. graded
alg. of the rad. filt. of $\text{proj } B$.

• In part, we get Ext quiver,
Cartan invariants, ...

Conj: (Auslander-Reiten)

If A, B are stably equiv., ~~then~~
then they have same number of
non-proj simples.

~~Let~~ Consider a 2-block with defect group C_2^3 and inertial quotient C_7 .
[Puing \Rightarrow stab. eq. to $k[\mathbb{F}_8 \rtimes \mathbb{F}_8^\times]$]

Thm (Landrock) (1981)

Then either:

(i) 8 ord chars, 7 simple mod

ω (ii) 5 ord chars, 4 simple mod.

Thm (Kessar, Koshitani, Lückehmann) 2010

CFSG \Rightarrow not (ii).

But what about algs that are not blocks?

$k[\mathbb{F}_8 \rtimes \mathbb{F}_8^\times]$ has 7 one-dim
 simples $1, 2, \dots, 7$

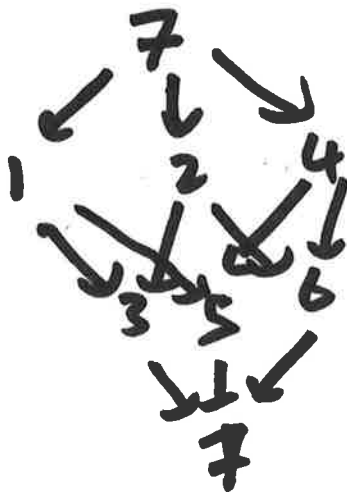
projectives:

\mathbb{F}_8^\times acts on $\frac{\text{rad}(k\mathbb{F}_8)}{\text{rad}^2(k\mathbb{F}_8)}$

with eigenvalues ζ, ζ^2, ζ^4 ($\zeta^7=1$)

Let $\pi_1, \pi_2, \pi_4 \in \text{rad}(k\mathbb{F}_8)$
 be corresp. eigenvectors

$P(7) =$



We're looking for 4 X_i 's.

But if we find a SMS stable under $\text{Gal}(\mathbb{F}_8/\mathbb{F}_2)$ then we have 2 orbits

So we just need to find essentially only two modules, one Galois-stable.