# SIMPLE MODULES & STABLE CATEGORIES

Suppose
$$F : D^b(B) \xrightarrow{\sim} D^b(A)$$
simples $S_1, \ldots, S_n \longleftrightarrow X_1, \ldots, X_n$

Then

$$\circledast \begin{cases} \mathrm{Hom}(X_i, X_j) = \begin{cases} k & i=j \\ 0 & i \neq j \end{cases} \\ \mathrm{Hom}(X_i, X_j[t]) = 0 \quad (t<0) \\ \{X_i\} \text{ gen}^s\ D^b(A) \text{ as a} \\ \text{triang cat.} \end{cases}$$

$\circledast$ = "simple minded collection".

<u>THM</u> If $A$ symm (i.e. $A \cong DA$ as bimodules) then any simple m. coll. comes from a derived equiv.

__Example__ char $k = 2$

$$A = kA_4 = k[\mathbb{F}_4 \rtimes \mathbb{F}_4^*]$$

has 3 one-dim simple modules $(\omega^3 = 1)$ 1, 2, 3 according to power of $\omega$ that a gen of $\mathbb{F}_4^*$ acts by.

Proj$^s$

$$\begin{smallmatrix} & 3 & \\ 1 & & 2 \\ & 3 & \end{smallmatrix}\ ,\quad \begin{smallmatrix} & 2 & \\ 3 & & 1 \\ & 2 & \end{smallmatrix}\ ,\quad \begin{smallmatrix} & 1 & \\ 2 & & 3 \\ & 1 & \end{smallmatrix}$$

$$X_1 = 3\ ,\quad X_2 = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\ ,\quad X_3 = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$$

$$\begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \to 1 \oplus \begin{smallmatrix} & 2 & \\ 3 & & 1 \\ & 2 & \end{smallmatrix} \to \begin{smallmatrix} & 2 & \\ 3 & & 1 \end{smallmatrix}$$

Suppose $A, B$ are self-inj with no simple proj$^s$, and $\exists$ exact

$$F : \text{mod-}B \to \text{mod-}A$$

s.t. $F$ induces an equiv.

$$F : \underline{\text{mod}}\text{-}B \to \underline{\text{mod}}\text{-}A$$

$$\begin{matrix}\text{simples} \\ S_1, \dots, S_n\end{matrix} \leftrightarrow X_1, \dots, X_n$$

Then

(**)

$$\underline{\mathrm{Hom}}(X_i, X_j) = \begin{cases} k & (i=j) \\ 0 & (i \neq j) \end{cases}$$

$\underline{\text{mod}}$-$A$ is the smallest subcat cont $\{X_i\}$ and closed under extensions

[i.e, for every simple $A$-mod $S$ $\exists$ proj $P$ s.t. $S \oplus P$ has a filt. with factors in $\{X_i\}$.]

(**) = "simple minded system".

Question:

Does every SMS come from a stable equiv?

If so, what can we say about A?

Linckelmann's Lemma: If a stab eq. of Morita type sends simples to simples then it is a Morita equiv.

So if A exists, it's unique.

jt with Rouquier:

Assume $\{X_i\}$ is SMS.

- For any non-proj. B-module M, there's a min. proj $P_M$ s.t. $M \oplus P_M$ has a filt with factors in $\{X_i\}$.
- If F exists, this reconstructs $F(F^{-1}(M))$ and the filt. induced by a comp. series of $F^{-1}(M)$.

- Applying to ~~[illegible]~~ $\Omega X_i$ (~~[illegible]~~) and sticking $X_i$ on top, we reconstruct a model of the Loewy series of proj B-mod^s.

- This gives us the assoc. graded alg. of the rad. filt. of ~~pc~~ B.

- In part, we get Ext quiver, Cartan invariants, ...

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<u>Conj</u>: (Auslander-Reiten)
If A, B are stably equiv., ~~then~~ then they have same number of non-proj simples.

Consider a 2-block with defect group $C_2^3$ and inertial quotient $C_7$.

[Puig $\Rightarrow$ stab. eq. to $k[\mathbb{F}_8 \rtimes \mathbb{F}_8^\times]$]

<u>Thm</u> (Landrock) (1981)

Then either:

(i) 8 ord chars, 7 simple mod

or (ii) 5 ord chars, 4 simple mod.

<u>Thm</u> (Kessar, Koshitani, Linckelmann 2010)

CFSG $\Rightarrow$ <u>not</u> (ii).

But what about algs that are <u>not</u> blocks?

$k[\mathbb{F}_8 \rtimes \mathbb{F}_8^\times]$ has 7 one-dim simples $1, 2, \ldots, 7$

<u>projectives</u>:

$\mathbb{F}_8^\times$ acts on $\dfrac{\mathrm{rad}(k\mathbb{F}_8)}{\mathrm{rad}^2(k\mathbb{F}_8)}$

with eigenvalues $\zeta, \zeta^2, \zeta^4$ $(\zeta^7 = 1)$

Let $x_1, x_2, x_4 \in \mathrm{rad}(k\mathbb{F}_8)$ be corresp. eigenvectors

$$P(7) = \begin{tikzcd} & 7 \arrow[dl] \arrow[d] \arrow[dr] & \\ 1 \arrow[d] \arrow[dr] & 2 \arrow[dl] \arrow[dr] & 4 \arrow[dl] \arrow[d] \\ 3 \arrow[dr] & 5 \arrow[d] & 6 \arrow[dl] \\ & 7 & \end{tikzcd}$$

we're looking for 4 $X_i$'s.

But if we find a SMS stable under $\mathrm{Gal}(\mathbb{F}_8/\mathbb{F}_2)$ then we have 2 orbits

So we just need to find essentially only <u>two</u> modules, one Galois-stable.