Stratification and cosupport for finite group schemes

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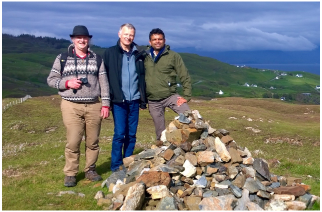


Joint work with BIK

Dave Benson,
Henning Krause,
Srikanth B. Iyengar

(left to right)

June 25, 2015

BensonFest, Isle of Skye

The party goes on: BensonFest II in Vancouver, BC
Summer School July 27-30, Conference August 1-5, 2016

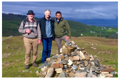
Summer school: G-finite group; k - field.
Classify *thick tensor ideals of* stmod kG.

This talk: k - field, G-finite group scheme over k.

Classify *localising tensor ideal subcategories of* StMod G.

$$\Longrightarrow$$

1. Classification of thick tensor ideals of stmod G for G a *finite group scheme* (corrected proof)
2. Classification of localising tensor ideal subcategories of StMod kG for G a *finite group* (new proof, originally due to



, Ann. of Math. 174 (2011))

Stable module category

Λ = finite dimensional Hopf k-algebra

StMod Λ Objects: (all) Λ-modules
Morphisms: $\underline{\mathrm{Hom}}(M, N) = \frac{\mathrm{Hom}_\Lambda(M,N)}{\mathrm{PHom}_\Lambda(M,N)}$
stmod Λ - finite dimensional Λ-modules

Λ is Frobenius $\Rightarrow$ StMod Λ is a tensor triangulated category

Localising subcategory $\mathcal{C} \subset$ StMod Λ: full triangulated subcategory closed under arbitrary direct sums.
Thick subcategory $\mathcal{C} \subset$ stmod Λ: full triangulated subcategory closed under direct summands.
Tensor ideal: $M \in \mathcal{C} \Rightarrow N \otimes M \in \mathcal{C}$ for any N.

Classify tensor ideal localising subcategories of StMod Λ

Problem: $H^*(\Lambda, k)$ is not known to be finitely generated!

STABLE MODULE CATEGORY

char k = p > 0

Λ = finite dimensional *cocommutative* Hopf k-algebra

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Why cocommutative? Geometric interpretation.

Definition

A finite group scheme G over a field k is a functor

$$G\colon \text{ comm. } k\text{-algebras} \longrightarrow \text{groups}$$

represented by a finite dimensional commutative Hopf k-algebra $k[G]$.

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$k[G]$ is a finite dimensional commutative Hopf k-algebra $\Rightarrow$ $kG := k[G]^* = \mathrm{Hom}_k(k[G], k)$ is a finite dimensional, cocommutative Hopf k-algebra, the group algebra of G

$$\left\{ \begin{array}{c} \text{finite group} \\ \text{schemes} \\ G \end{array} \right\} \sim \left\{ \begin{array}{c} \text{finite dimensional} \\ \text{cocommutative} \\ \text{Hopf algebras} \\ kG \end{array} \right\}$$

Representations of G over k $\longleftrightarrow$ kG-modules

EXAMPLES

- Finite groups. kG is the group algebra
- Restricted Lie algebras. For $\mathcal{G}$ - algebraic group (GL_n, SL_n, Sp_{2n}, SO_n), $\mathfrak{g} = \mathrm{Lie}\,\mathcal{G}$.

$$\mathfrak{u}(\mathfrak{g}) = U(\mathfrak{g})/\langle x^p - x^{[p]}\rangle$$

  restricted enveloping algebra, a finite dimensional cocommutative Hopf algebra
- Frobenius kernels

$$G = \mathcal{G}_{(r)} = \mathrm{Ker}\{\mathcal{G} \xrightarrow{F^r} \mathcal{G}\}$$

  Frobenius kernels are *connected* ($k[G]$ is local).

$$\left\{ \begin{matrix} \text{Restricted} \\ \text{Lie algebras} \\ \mathrm{Lie}\,\mathcal{G} \end{matrix} \right\} \sim \left\{ \begin{matrix} \text{Connected finite group} \\ \text{schemes of height 1} \\ \mathcal{G}_{(1)} \end{matrix} \right\}$$

$$\mathfrak{u}(\mathrm{Lie}\,\mathcal{G}) \cong k\mathcal{G}_{(1)}$$

$H^*(G,k) := H^*(kG,k)$.
To apply geometric methods, work with $\operatorname{Proj} H^*(G,k)$

Theorem (Friedlander-Suslin, 1997)

Let G be a finite group scheme over a field k, and M be a finite dimensional representation of G. Then $H^*(G,k)$ is a finitely generated k-algebra, and $H^*(G,M)$ is a finite module over $H^*(G,k)$.

Theorem (Suslin-Friedlander-Bendel, 1997)

Let $\mathfrak{g}$ be a restricted Lie algebra. Then

$$\operatorname{Spec} H^*(\mathfrak{g},k) \cong \mathcal{N}_p(\mathfrak{g}) := \{x \in \mathfrak{g} \mid x^{[p]} = 0\}$$

For $\mathcal{G}$ a connected reductive algebraic group, $\mathcal{N}_p(\operatorname{Lie}\mathcal{G})$ is irreducible! Contrast with Quillen stratification theorem for $H^*(G,k)$ for finite groups.
Moral: there is no family of abelian subgroup schemes controlling the behavior of $H^*(G,k)$ or $\operatorname{stmod} kG$.

Theorem (-P., 2015)

Let G be a finite group scheme over a field k. There is a one-to-one correspondence

$$\left\{ \begin{array}{c} \textit{Localising tensor} \\ \textit{ideal subcategories} \\ \textit{of } \operatorname{StMod} kG \end{array} \right\} \sim \left\{ \begin{array}{c} \textit{Subsets of} \\ \operatorname{Proj} H^*(G,k) \end{array} \right\}$$

which restricts to

$$\left\{ \begin{array}{c} \textit{Thick tensor ideal} \\ \textit{subcategories of } \operatorname{stmod} kG \end{array} \right\} \sim \left\{ \begin{array}{c} \textit{Subsets of } \operatorname{Proj} H^*(G,k) \\ \textit{closed under specialization} \end{array} \right\}$$

$$\mathcal{C}_V = \{M \in \operatorname{StMod} kG \mid \operatorname{supp} M \subset V\} \longleftarrow V \subset \operatorname{Proj} H^*(G,k)$$

$$\mathcal{C} \longrightarrow V = \bigcup_{M \in \mathcal{C}} \operatorname{supp} M$$

Ingredients of the proof

Precursors/motivation: Devinatz-Hopkins-Smith (stable homotopy theory), Hopkins, Neeman, Thomason (CA, AG), Benson-Carlson-Rickard (Finite groups)

- Need theory of supports!
- And cosupports
- In fact, one theory is not enough; need two:
  - BIK theory of local cohomology functors (Rickard idempotents)
  - π-supports and π-cosupports
- Detection of projectivity by π-supports (generalized Dade's lemma)
- Comparison of Koszul objects ($\sim$ Carlson modules) for closed and generic points on $\operatorname{Proj} H^*(G, k)$

BIK SUPPORT / COSUPPORT

$X = \operatorname{Proj} H^*(G, k)$

$$\mathfrak{p} \in X \mapsto \Gamma_{\mathfrak{p}} k \in \operatorname{StMod} kG$$

- "Rickard idempotent", a universal module with respect to the classical support variety theory based on the action of $H^*(G, k)$ on $H^*(S, M)$ (as appeared in J. Carlson's talk).

For M a kG-module,

$$\operatorname{supp} M := \{\mathfrak{p} \in X \mid \Gamma_{\mathfrak{p}} k \otimes M \text{ is not projective}\}$$

$$\operatorname{cosupp} M := \{\mathfrak{p} \in X \mid \operatorname{Hom}_k(\Gamma_{\mathfrak{p}} k, M) \text{ is not projective}\}$$

$M \in \operatorname{stmod} kG \implies \operatorname{supp} M = \operatorname{cosupp} M =$"classical" support variety.
Good properties: "two out of three", direct sums, shifts, detection ($\operatorname{supp} M = \emptyset \Rightarrow M \cong 0$, that is, M is projective).
Lack: good behavior w.r.t tensor products and Homs.

π-POINTS

Definition

A π**-point** α of a finite group scheme G defined over field extension K/k is a flat map of algebras

$$\begin{array}{ccc} K[t]/t^p & \overset{\alpha}{\dashrightarrow} & KG := kG \otimes_k K \\ & \searrow \; \nearrow & \\ & KU & \end{array}$$

which factors through some unipotent abelian subgroup scheme $U \subset G_K$.

A finite group scheme U is unipotent if KU is a local algebra (unipotent finite groups = p-groups).
The map $KU \to KG$ is a map of Hopf algebras, the other two are just maps of algebras.

From π-points to points on $\operatorname{Proj} H^*(G,k)$

$$\boxed{\alpha : K[t]/t^p \to KG} \rightsquigarrow H^*(G_K, K) \xrightarrow{\alpha^*} H^*(K[t]/t^p, K) \rightsquigarrow$$
$$\operatorname{Ker} \alpha^* \cap H^*(G,k) \rightsquigarrow \boxed{\mathrm{pt} \in \operatorname{Proj} \mathrm{H}^*(\mathrm{G}, \mathrm{k})}$$

Some π - points $\rightsquigarrow$ same point on $\operatorname{Proj} H^*(G,k)$

$$\alpha^* : \mathsf{Mod}\, kG \longrightarrow \mathsf{Mod}\, K[t]/t^p\ , \quad M \mapsto \alpha^*(M_K).$$

$$\alpha : K[t]/t^p \longrightarrow KG\ ,\ \beta : L[t]/t^p \longrightarrow LG$$

$$\{\ \alpha \sim \beta\ \} \quad \overset{\text{def}}{\Longleftrightarrow} \left\{ \begin{array}{c} \forall\ \text{(fin. dim.)}\ kG-\text{module}\ M, \\ \alpha^*(M_K)\ \text{is free} \iff \beta^*(M_L)\ \text{is free} \end{array} \right\}$$

$$\Pi(G) := \frac{\langle \pi\text{−points}\rangle}{\sim}$$

Theorem (Friedlander-P, 2007)

There is a natural homeomorphism $\Pi(G) \cong \operatorname{Proj} H^*(G,k)$.

π-SUPPORT AND π-COSUPPORT

Let M be a kG-module.

$$\pi\text{-}\operatorname{supp} M := \{[\alpha] \in \Pi(G) \mid \alpha^*(M \otimes_k K) \text{ is not free } \}$$
$$\pi\text{-}\operatorname{cosupp} M := \{[\alpha] \in \Pi(G) \mid \alpha^*(\operatorname{Hom}_k(K, M)) \text{ is not free } \}$$

$\{\pi\text{-}\operatorname{supp} M\}$ for M finite dimensional kG-modules are precisely the closed sets in $\Pi(G)$.

Theorem (Friedlander-P.'07, BIKP'15)

Let M, N be kG-modules.

[Tensor product formula]
$\pi\text{-}\operatorname{supp} M \otimes_k N = \pi\text{-}\operatorname{supp} M \cap \pi\text{-}\operatorname{supp} N$

[Function object formula]
$\pi\text{-}\operatorname{cosupp} Hom_k(M, N) = \pi\text{-}\operatorname{supp} M \cap \pi\text{-}\operatorname{cosupp} N$

Two support theories: BIK (co)support and π-(co)support.
Identify them $\Longrightarrow$ prove classification for $\operatorname{StMod} kG$

Short (conceptual and elegant) route for finite groups.

Theorem

Let G be a finite group, and M be a kG-module. Then
$\pi\text{-}\operatorname{cosupp}(M) = \emptyset \iff M$ *is projective.*

Proof: analogue of Dade's lemma for elementary abelian p-groups + Chouinard's theorem.

Theorem

Let G be a finite group, and M be a kG-module. Then
$\pi\text{-}\operatorname{cosupp} M = \operatorname{cosupp} M$
$\pi\text{-}\operatorname{supp} M = \operatorname{supp} M$

"Local to Global Principle" (BIK theory) $\Longrightarrow$ to classify localising tensor ideals in $\operatorname{StMod} kG$ it suffices to prove

Minimality: For any $\mathfrak{p} \in \operatorname{Proj} H^*(G, K)$,
$\Gamma_{\mathfrak{p}}(\operatorname{StMod} kG) :\ = \{M \in \operatorname{StMod} kG \mid \operatorname{supp} M \subseteq \mathfrak{p}\}$ is a minimal tensor ideal localising subcategory.

Theorem

Let G be a finite group. Then $\Gamma_{\mathfrak{p}}(\operatorname{StMod} kG)$ is minimal for any $\mathfrak{p} \in \operatorname{Proj} H^*(G, k)$.

Proof. It suffices to show that for any $M, N \ncong 0$ in $\Gamma_{\mathfrak{p}}(\operatorname{StMod} kG)$, $\operatorname{Hom}_k(N, M) \ncong 0$.

1. $M \ncong 0 \Longrightarrow \operatorname{End}_k(M) \ncong 0$
2. $\emptyset \neq \operatorname{cosupp}(\operatorname{End}_k(M)) = \operatorname{supp} M \cap \operatorname{cosupp} M \Longrightarrow \mathfrak{p} \in \operatorname{cosupp} M$
3. $\operatorname{cosupp}(\operatorname{Hom}_k(N, M)) = \operatorname{supp} N \cap \operatorname{cosupp} M = \mathfrak{p} \Longrightarrow \operatorname{Hom}_k(N, M) \ncong 0$
4. The end!

Classification for finite group schemes

Detection of projectivity by π-cosupport for *arbitrary finite group scheme* is problematic.

Theorem (Super[a] generalized Dade's lemma)

[a] "super" = "big and powerful"

Let G be a finite group scheme, and M be a kG-module. Then M is projective if and only if for every field extension K/k and every flat algebra map $\alpha : K[t]/t^p \to KG$, the $K[t]/t^p$-module $\alpha^*(M \otimes_k K)$ is projective.

Benson-Carlson-Rickard, Bendel, P., Benson-Iyengar-Krause-P.

Important: holds for infinite-dimensional modules.

Equivalent formulation:

$$M \cong 0 \text{ in } \operatorname{StMod} kG \iff \pi\text{-}\operatorname{supp} M = \varnothing$$

Theorem

For any finite group scheme G, and any kG-module M,
$\pi\text{-}\operatorname{supp} M = \operatorname{supp} M$

To prove that $H^*(G,k)$ "stratifies" $\operatorname{StMod} kG$ (which implies classification), it suffices to prove minimality of $\Gamma_{\mathfrak{p}}(\operatorname{StMod} kG)$.

Theorem

Let $\mathfrak{m} \in \operatorname{Proj} H^*(G,k)$ *be a closed point. Then* $\Gamma_{\mathfrak{m}}(\operatorname{StMod} kG)$ *is minimal.*

Proof. Formal from three ingredients:

- $\pi\text{-}\operatorname{supp} = \operatorname{supp}$
- $\pi\text{-}\operatorname{supp}$ detection of projectivity
- Function Object Formula for $\pi\text{-}\operatorname{cosupp}$

$\mathfrak{p} \in X = \operatorname{Proj} H^*(G, k)$

$$\begin{array}{ccc} \mathfrak{m} & \in & X_K = \operatorname{Proj} H^*(G_K, K) \\ \downarrow & & \downarrow \\ \mathfrak{p} & \in & X = \operatorname{Proj} H^*(G, k) \end{array}$$

$\mathfrak{m}$ is a closed point in X_K "lying over" $\mathfrak{p}$.

Theorem (Reduction to closed points)

$\Gamma_{\mathfrak{p}} k \in \mathrm{Loc}^{\otimes}(\Gamma_{\mathfrak{m}} K{\downarrow_G})$. Equivalently, $\Gamma_{\mathfrak{m}} K{\downarrow_G}$ "builds" $\Gamma_{\mathfrak{p}} k$.

Main ingredient: explicit comparison of Koszul objects (= Carlson modules): $\Omega^d (K/\!\!/\mathfrak{m}){\downarrow_G} \simeq (k/\!\!/\mathfrak{p})_{\mathfrak{p}}$.

Corollary

$\Gamma_{\mathfrak{p}}$ is minimal for any $\mathfrak{p} \in \operatorname{Proj} H^*(G, k)$. Hence, $H^*(G, k)$ stratifies $\operatorname{StMod} kG$ and classification theorem holds.

Applications

- π- cosupp $=$ cosupp
- π- cosupp detects projectivity of kG-modules

Quiz! What does Dave do once he is done with stratifying?

1. Drinks beer
2. Drinks whisky
3. Drinks Spanish red wine
4. Starts costratifying

And the correct answer is .... 3 and 4 ...no, wait, it is "all of the above".

Theorem

Let G be a finite group scheme. Then $H^*(G,k)$ "costratifies" $\operatorname{StMod} kG$. Hence, there is one-to-one correspondence between colocalising Hom-closed subcategories of $\operatorname{StMod} kG$ and subsets of $\operatorname{Proj} H^*(G,k)$, given by cosupport.

HAPPY BIRTHDAY, DAVE!