Mathematics for Actuarial Science 1

1. Given that
\[ cx = \sqrt{\left( ax^2 - b \right) \left( \frac{c}{d} \right)} \]
express \( x \) in terms of \( a, b, c, \) and \( d \).

2. If \( x = p + \sqrt{q} \) where \( p \) and \( q \) are rational, show that \( x^2 \) and \( x^3 \) are of the form \( P + Q \sqrt{q} \), where \( P \) and \( Q \) are rational.

3. Verify that \( z = (4 + \sqrt{15})^3 + (4 - \sqrt{15})^3 \) satisfies
\[ z^3 - 3z - 8 = 0. \]

4. Expand \((2-3x)^5\), arranging your answer in ascending powers of \( x \) with integer coefficients.

5. Given that \((1+2x)^2 = 1 + Ax + Bx^2 + Cx^3 + \cdots\), find the values of \( A, B, \) and \( C \).

6. Show that
\[ \left(x + \frac{1}{x}\right)^3 + \left(x - \frac{1}{x}\right)^3 = 2x^3 + \frac{6}{x}. \]

7. Calculate the value of the term independent of \( x \) in the expansion of \((x^2 - \frac{3}{4})^6\).

8. Show that
\[ \frac{(2n)!}{n!} = 2^n \cdot 1.3.5 \cdots (2n - 1). \]

9. (**) Write down the general term in the expansion of \((1 + x)^n\). Use the identity
\[ (1 + x)^m(1 + x)^n = (1 + x)^{m+n} \]
to prove that
\[ a'C_0 + a'C_{-1}C_1 + a'C_{-2}C_2 + \cdots + a'C_r = aC_r. \]

10. Solve the equation \( \frac{1}{x} + \frac{1}{x^2} = 2 \).

11. Given that \( \alpha \) and \( \beta \) are roots of the equation \( x^2 + 3x - 6 = 0 \), find a quadratic equation with integer coefficients whose roots are \( \frac{\alpha}{n} \) and \( \frac{\beta}{n} \).

12. Find the set of values of \( k \) for which the equation \( x^2 + kx + (3-k) = 0 \) has real roots.

In the case when \( k = 5 \), the roots of the equation are \( \alpha \) and \( \beta \). Without calculating the values of \( \alpha \) and \( \beta \), find

(a) the value of \( \alpha^2 + \beta^2 \);

(b) a quadratic with roots \( \alpha^2 + 3\beta \) and \( \beta^2 + 3\alpha \).

13. Divide \( x^6 + 5x^5 + 11x^4 + 13x^3 - 3x^2 - 8x + 5 \) by \( x^2 + 2x + 5 \).

14. Show that \( x - 4 \) is a factor of \( f(x) = x^6 - 8x^2 + 29x - 52 \). Factorise \( f(x) \) and show that the equation \( f(x) = 0 \) has only one real root.

15. Use the remainder theorem to find a factor of \( f(x) = 2x^3 - 9x^2 + 7x + 6 \), and hence factorise \( f(x) \) into its linear factors.

16. The function \( f(x) \) is given by \( f(x) = x^3 + ax^2 - 4x + b \), where \( a \) and \( b \) are constants.

Given that \( x - 2 \) is a factor of \( f(x) \) and that there is a remainder of 6 when \( f(x) \) is divided by \( x - 1 \), find the values of \( a \) and \( b \).

17. Show that
\[ \frac{1}{1+x} + \frac{8}{2-x} + \frac{12}{(2-x)^2} = \frac{kx^2}{(1+x)(2-x)^2} \]
where \( k \) is an integer to be determined.

18. Express
\[ \frac{1 + 3x^2}{(1+x)^2(1+3x)} \]
in partial fractions.

19. Express
\[ \frac{1 - 2x + 5x^2}{(1-2x)(1+x^2)} \]
in partial fractions.

20. (**) Express \( x^4 - 4x^2 + 16 \) in the form
\[ (x^2 + Ax + B)(x^2 + Cx + D) \]
where \( A, B, C, \) and \( D \), are real constants. Hence express
\[ \frac{1}{x^4 - 4x^2 + 16} \]
in partial fractions.

21. (**) Express
\[ \frac{x^5 - 1}{x^2(x^3 + 1)} \]
in partial fractions.