Mathematics for Actuarial Science 6

1. Calculate
   (a) \( \int x \ln x \, dx \)  
   (b) \( \int x (\ln x)^2 \, dx \).

2. Calculate
   (a) \( \int \sqrt{3x + 5} \, dx \)  
   (b) \( \int (3x + 8) e^x \, dx \)  
   (c) \( \int 3x \sqrt{3x + 5} \, dx \).

3. (a) Given that \( 2y = x - \sin x \cos x \), show that \( \frac{dy}{dx} = \sin^2 x \).
   (b) Hence find \( \int x \sin^2 x \, dx \).

4. Calculate
   \( \int_0^\pi x^2 \cos 3x \, dx \).

5. Let \( I_n \) stand for the integral \( \int x^n e^x \, dx \). Use integration by parts to give a formula relating \( I_n \) to \( I_{n-1} \). Use this result to find \( I_4 \).

6. The curve with equation \( y = e^{3x} + 1 \) meets the line \( y = 8 \) at the point \( (b, 8) \).
   (a) Find \( b \), giving your answer in terms of natural logarithms.
   (b) Show that the area of the finite region enclosed by the curve with equation \( y = e^{3x} + 1 \), the \( x \)-axis, the \( y \)-axis, and the line \( x = b \), is \( 2 + \frac{1}{3} \ln 7 \).

7. The graph of \( y = x(4 - x^2) \) is illustrated below for \( x \geq 0 \). Find the exact value of \( k \) for which the areas above and below the \( x \)-axis are equal.

8. Curves \( C \) and \( D \) have equations \( y = \frac{1}{4} \) and \( y = kx^2 \) respectively, where \( k \) is a constant. The curves intersect at the point \( P \), whose \( x \)-coordinate is \( \frac{1}{2} \).
   (a) Determine the value of \( k \).
   (b) Find the gradient of \( C \) at \( P \).
   (c) Calculate the area of the finite region bounded by \( C \), \( D \), the \( x \)-axis, and the line \( x = 2 \).

9. Simplify \( \tan(\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{4}) \).

10. Solve the equation \( \sin^{-1} \left( \frac{x}{x + 1} \right) + 2 \tan^{-1} \left( \frac{1}{x + 1} \right) = \frac{\pi}{2} \).

11. Calculate \( \int \frac{3}{2x^2 + 5} \, dx \).