AS1051: Mathematics

0. Introduction

The aim of this course is to review the basic mathematics which you have already learnt during A-level, and then develop it further. You should find it almost entirely familiar, with only the occasional topic of new material.

However, even for those parts which are familiar there will be aspects which will be treated differently at university level. Most importantly, you will be expected to *know* key facts and formulas — you should not expect to have a formula sheet at your disposal. Thus, for example, you will be expected to memorise all the standard integrals and derivatives, the trigonometric identities, etc.

You will also notice that the pace of university mathematics is much faster than at school. This will certainly be true of the revision of A-level material, but will also extend to the new material. In part this will be because there will be fewer worked examples; you will be expected to practise calculations by yourself. Also, if you do not keep up to date, the speed of the course will make it hard for you to catch up.

In all courses it is important that you attempt the exercise sheets. These will not be marked, but without working through them you are very unlikely to perform well in the final exams. Tutors are very pleased when students ask questions about material they do not understand — you should make full use of them!

There are several books recommended on the course webpage. My personal recommendation would be the A-level textbook Bostock and Chandler: Pure Mathematics (possibly together with Further Pure Mathematics by the same authors with Rourke).

An alternative would be Jordan and Smith: Mathematical Techniques. This goes more quickly through the basic material in this course, but goes on to cover more advanced topics that you will see in many of your first and second year modules.

1. Arithmetic

In this chapter we will review the basic algebraic manipulations which should already be familiar. First we introduce the main classes of numbers.

1.1 Numbers

Most basic are the natural numbers, $\mathbb{N}$, which consist of the positive whole numbers $1, 2, 3, \ldots$ (Some textbooks include 0 as a natural number.) Note in passing that positive means > 0, and negative means < 0. To talk about numbers ≥ 0 we say non-negative.

The integers $\mathbb{Z}$ consist of all whole numbers $0, \pm 1, \pm 2, \ldots$

An integer a is divisible by another (non-zero) integer b if there exists a third integer c such that $a = bc$. In this case we call b a divisor of a. An integer p is prime if $p > 1$ and p has no positive divisors except 1 and p.

Although easy to define, integers are hard to completely understand. For example, we do not have a formula for determining quickly whether a given number is prime.

Primes are important because of

Theorem 1.1.1: (The fundamental theorem of arithmetic)
Every positive integer has a unique prime factorisation.

Note that this results says two things: there is a factorisation as a product of primes, and it is unique.

Example 1.1.2: $2,522,520 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times 7 \times 11 \times 13$.

Given two non-zero integers m and n we define their highest common factor $\text{hcf}(m, n)$ to be the largest divisor of m and n, and the least common multiple $\text{lcm}(m, n)$ to be the smallest positive integer divisible by m and n.

Example 1.1.3: If $m = 60 = 2 \times 2 \times 3 \times 5$ and $n = 70 = 2 \times 5 \times 7$, then $\text{hcf}(m, n) = 10$ and $\text{lcm}(m, n) = 420$.

The rational numbers , $\mathbb{Q}$, consist of all numbers of the form $r = \frac{p}{q}$ where p and q are integers with $q \neq 0$. Note that there are equivalent forms of a rational number:

$$\frac{p}{q} = \frac{s}{t} \quad \text{if and only if} \quad pt = qs.$$

Example 1.1.4: $\frac{3}{5} = \frac{9}{15}$ as $3 \times 15 = 5 \times 9$.

We usually simplify fractions to the form $r = \frac{p}{q}$ where $\text{hcf}(p, q) = 1$. (Integers with $\text{hcf}(p, q) = 1$ are called coprime.)

If we imagine numbers as making up a line, then in any given segment there are infinitely many rationals. However, not every number is rational. For example, $\sqrt{2}$ is not rational (this will be proved later in the course). We will call such numbers irrational.

The real numbers, $\mathbb{R}$, consist of all rational and irrational numbers.

Note that we have not given a precise definition of $\mathbb{R}$, as we have not really said what irrational numbers are. This is because $\mathbb{R}$ is rather hard to define! It took most of the nineteenth century for mathematicians to come up with a definition which actually reflected the properties of real numbers that we 'know' that we require.

Remark 1.1.5: Never approximate fractions, square roots, etc., by decimals, unless you are specifically asked for an approximate answer.

1.2 Laws of indices

We are already familiar with basic exponents:

$$a^n = \begin{cases} a \times \cdots \times a \;\; n \text{ times} & \text{if } n \in \mathbb{N} \\ 1 & \text{if } n = 0 \\ \frac{1}{a^{-n}} & \text{if } n \in \mathbb{Z} \text{ and } n < 0. \end{cases}$$

Here $\in$ means "is an element of". These satisfy:

$$\begin{aligned} a^n \times a^m &= a^{n+m} & (a^n)^m &= a^{nm} \\ (ab)^n &= a^n b^n & (a/b)^n &= (a^n)/(b^n) \end{aligned} \tag{1}$$

for all $a, b \neq 0$.

For $a > 0$ we want to define a^r for all $r \in \mathbb{Q}$. This can even be done for all $r \in \mathbb{R}$, but we will not do so here.

First, for $n \in \mathbb{N}$, let $x = a^{\frac{1}{n}}$ be the positive real x such that $x^n = a$. We also write $\sqrt[n]{a}$ for $a^{\frac{1}{n}}$. Such an x always exists and is unique.

Now we can define a^r for any $r = \frac{p}{q}$ with $p, q \in \mathbb{N}$ and q non-zero by

$$a^{\frac{p}{q}} = (a^{\frac{1}{q}})^p \quad \text{and} \quad a^{-\frac{p}{q}} = \left(a^{\frac{p}{q}}\right)^{-1}.$$

If $r = \frac{p}{q} = \frac{s}{t}$ then this gives the same answer as using $(a^{\frac{1}{t}})^s$ also, so this is well defined.

We still have the properties in equation (1) for rational powers.

Example 1.2.1: Evaluate $9^{-\frac{1}{2}} + \left(\frac{16}{81}\right)^{\frac{5}{4}}$.

$$9^{-\frac{1}{2}} + \left(\frac{16}{81}\right)^{\frac{5}{4}} = \frac{1}{3} + \left(\frac{2}{3}\right)^5 = \frac{1}{3} + \frac{32}{243} = \frac{113}{243}.$$

Example 1.2.2: Prove that $\sqrt{3 + 2\sqrt{2}} = 1 + \sqrt{2}$.

Let $x = 1 + \sqrt{2}$. Then $x > 0$ and

$$x^2 = (1 + \sqrt{2})^2 = 1 + 2\sqrt{2} + 2 = 3 + 2\sqrt{2}.$$

1.3 The binomial theorem

For $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$ we have

$$\begin{aligned} (a+b)^n = \;& a^n + na^{n-1}b + \tfrac{n(n-1)}{2}a^{n-2}b^2 + \cdots \\ & \cdots + \tfrac{n(n-1)(n-2)\cdots(n-r+1)}{2 \times 3 \times \cdots \times r}a^{n-r}b^r + \cdots + b^n. \end{aligned}$$

We write $a!$ for $a(a-1)\cdots 3 \times 2 \times 1$. Then the coefficient of $a^{n-r}b^r$ above can be written as

$$\frac{n!}{r!(n-r)!}$$

which we denote by ${}_nC_r$ or $\binom{n}{r}$ (and pronounce "n choose r").

Example 1.3.1:

$$\begin{aligned} (a+b)^2 &= a^2 + 2ab + b^2 \\ (a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3. \end{aligned}$$

We can arrange the ${}_nC_r$ into a triangle, called Pascal's triangle.

$$\begin{array}{ccccccc} & & {}_1C_0 & & {}_1C_1 & & \\ & {}_2C_0 & & {}_2C_1 & & {}_2C_2 & \\ {}_3C_0 & & {}_3C_1 & & {}_3C_2 & & {}_3C_3 \\ & & & \vdots & & & \end{array}$$

$$\begin{array}{ccccccccc} & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ & & & & \vdots & & & & \end{array}$$

From the triangle it appears that each entry is the sum of the two entries above it — but an example is not a proof! However, this does turn out to be true.

Theorem 1.3.2: *For $n \geq 1$ and $1 \leq r \leq n$ we have*

$${}_nC_{r-1} + {}_nC_r = {}_{n+1}C_r.$$

Proof:

$$\begin{aligned} {}_nC_{r-1} + {}_nC_r &= \tfrac{n!}{(r-1)!(n-r+1)!} + \tfrac{n!}{r!(n-r)!} \\ &= \tfrac{n!}{r!(n-r+1)!}[r + (n-r+1)] = \tfrac{(n+1)!}{r!(n-r+1)!} = {}_{n+1}C_r. \end{aligned}$$

$\square$

Here the box at the end denotes the end of the proof.

Example 1.3.3: Find the term independent of x in the expansion of

$$\left(3x - \frac{5}{x^3}\right)^8.$$

Here $n = 8$, $a = 3x$, and $b = -5/x^3$.
The general term is

$$\frac{8!}{r!(8-r)!}(3x)^{8-r}\left(\frac{-5}{x^3}\right)^r = \frac{8!}{r!(8-r)!}3^{8-r}(-5)^r x^{8-4r}.$$

The power of x in this term is zero when $r = 2$, and so the required term is

$$\frac{8!}{2!6!}3^6(-5)^2 = 700 \times 3^6.$$

1.4 Permutations and combinations

Suppose we have a collection of n distinct objects. We can ask how many ways we can choose r objects from them if

- we do care what order we choose them in;
- we do not care what order we choose them in.

The first case is called the number of permutations and the second the number of combinations.

Example 1.4.1: From $1, 2, 3$ we have six permutations of two elements

$$1,2 \quad 1,3 \quad 2,1 \quad 2,3 \quad 3,1 \quad 3,2$$

and the following three combinations

$$1,2 \quad 1,3 \quad 2,3.$$

In general the number of permutations of r objects from a set of n distinct objects is given by

$$_nP_r = \frac{n!}{(n-r)!}$$

and the number of combinations is just $_nP_r/_rP_r$, which equals

$$_nC_r = \frac{n!}{r!(n-r)!}.$$

Note that this is the same as the coefficient in the binomial theorem.

1.5 Polynomials

A polynomial of degree n in x is a function $p(x)$ of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots\cdots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are constants with $a_n \neq 0$.

We call degree 2 polynomials quadratic, degree 3 cubic etc.

Traditionally we write quadratics in the form

$$ax^2 + bx + c.$$

To complete the square we write a quadratic in the form

$$a((x+d)^2 + e)$$

for some constants a, d, and e. In this case the roots (if they exist) are given by

$$x = -d \pm \sqrt{-e}$$

and if a is positive (respectively negative) then the minimum (respectively maximum) occurs at $x = -d$, and equals ae.

Example 1.5.1: We will complete the square for the following quadratic.

$$\begin{aligned} f(x) &= 3x^2 + 2x - 4 \\ &= 3\left(x^2 + \tfrac{2x}{3} - \tfrac{4}{3}\right) \\ &= 3\left(\left(x + \tfrac{1}{3}\right)^2 - \tfrac{13}{9}\right) \end{aligned}$$

has roots $x = -\frac{1}{3} \pm \sqrt{\frac{13}{9}}$ and a minimum at $x = -\frac{1}{3}$ of $-\frac{13}{3}$.

It is well known that the roots of $ax^2 + bx + c$ are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

So we have

- two distinct roots if $b^2 - 4ac > 0$,
- one root if $b^2 - 4ac = 0$,
- no roots if $b^2 - 4ac < 0$.

If we denote the roots by α and β then we have

$$f(x) = a(x-\alpha)(x-\beta) = ax^2 + bx + c$$

and so

$$a(x^2 - (\alpha+\beta)x + \alpha\beta) = ax^2 + bx + c.$$

From this we deduce that

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}.$$

Similar formulas can be deduced for cubics, quartics, etc.

Example 1.5.2: If the roots of $x^2 + 5x + 3 = 0$ are α and β, find the quadratic equation with roots α^3 and β^3.

We have $\alpha + \beta = -5$ and $\alpha\beta = 3$. So

$$\begin{aligned} \alpha^3 + \beta^3 &= (\alpha+\beta)^3 - 3\alpha^2\beta - 3\alpha\beta^2 \\ &= -125 - 3\alpha\beta(\alpha+\beta) \\ &= -125 - 9(-5) = -80. \end{aligned}$$

and $\alpha^3\beta^3 = (\alpha\beta)^3 = 27$. Thus the required equation is

$$x^2 + 80x + 27 = 0.$$

Returning to general polynomials, we can easily add and multiply them to form new polynomials. However, $p(x)/q(x)$ is not in general a polynomial.

Example 1.5.3: Let $p(x) = x^2 + 1$ and $q(x) = x - 2$. Then

$$p(x) + q(x) = x^2 + x - 1$$

and

$$p(x)q(x) = (x^2 + 1)(x - 2) = x^3 - 2x^2 + x - 2.$$

For $p(x)/q(x)$ we have

$$\begin{array}{r|l} & x \;+2 \\ \cline{2-2} x-2 & x^2+0x+1 \\ & x^2-2x \\ \cline{2-2} & \quad 2x+1 \\ & \quad 2x-4 \\ \cline{2-2} & \qquad\quad 5 \end{array}$$

and so $p(x)/q(x)$ equals

$$\frac{x^2+1}{x-2} = x + 2 + \frac{5}{x-2}.$$

Let $p(x)$ be a polynomial of degree n and divide $p(x)$ by $x - a$, where a is a constant:

$$\frac{p(x)}{x-a} = q(x) + \frac{r}{x-a}$$

where $q(x)$ is a polynomial and r is a constant, i.e.

$$p(x) = (x - a)q(x) + r.$$

From this we deduce

Theorem 1.5.4: *If $p(x)$ is a polynomial with $p(a) = r$ then*

$$p(x) = (x - a)q(x) + r$$

for some polynomial $q(x)$.

When $r \neq 0$ this is called the remainder theorem and when $r = 0$ it is called the factor theorem.

Example 1.5.5: Factorise

$$f(x) = x^3 - 7x^2 + 7x + 15.$$

We try some numbers: $f(0) = 15$, $f(1) = 16$, $f(-1) = 0$, and so $x + 1$ is a factor.

$$\begin{aligned} f(x) &= (x+1)(x^2 - 8x + 15) \\ &= (x+1)(x-3)(x-5). \end{aligned}$$

Fact: Every polynomial can be factorised into linear and/or quadratic terms.

If $p(x) = a_n x^n + \cdots + a_0$ has n distinct roots $x_1, \ldots, x_n$, then

$$p(x) = a_n(x - x_1)(x - x_2)\cdots(x - x_n).$$

Standard results (to be memorised):

$$\begin{aligned} x^2 - a^2 &= (x-a)(x+a) \\ x^3 - a^3 &= (x-a)(x^2 + ax + a^2) \\ x^n - a^n &= (x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1}) \end{aligned}$$

When n is odd we can get a formula for $x^n + a^n$ from the last one by replacing a by $(-a)$. However, there is no simple formula for the case n even.

The method of undetermined coefficients

If two polynomials are identical — i.e. are equal for every value of x — then the coefficients of like terms are equal.

Example 1.5.6: Find a, b, c, d such that

$$r^3 = ar(r-1)(r-2) + br(r-1) + cr + d.$$

Expanding we see that

$$\begin{aligned} r^3 &= a(r^3 - 3r^2 + 2r) + b(r^2 - r) + cr + d \\ &= ar^3 + (b - 3a)r^2 + (2a - b + c)r + d. \end{aligned}$$

Therefore $a = 1$, $b - 3a = 0$, $2a - b + c = 0$ and $d = 0$; i.e. $a = 1$, $b = 3$, $c = 1$, and $d = 0$.

1.6 Rational functions

A rational function is a function of the form $\frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials with $q(x)$ not identically zero. (That is, there is at least one value of x for which $q(x) \neq 0$.) For example

$$\frac{x^2 + 6x + 4}{x^2 - 5} \quad \text{and} \quad \frac{x+7}{3x^7 - 2x + 1}.$$

We can add or subtract rational functions just like we do ordinary fractions. We can also simplify them in the same way (by removing common factors from the top and bottom).

A proper rational function is one where the degree of the numerator is less than the degree of the denominator. Otherwise we say the function is improper. For example, the first fraction above is improper, the second proper.

There is a second way to simplify a rational function which has a product of factors in the denominator, using partial fractions. To do this to a fraction $p(x)/q(x)$ we use the following procedure:

Step 1: Simplify p/q to form a proper rational function.

Step 2: Factorise the denominator into linear and quadratic factors.

Step 3: If q has n factors, write the fraction as a sum of n terms using the following correspondence between factors of q and summands:

$$(x-a)^r \quad \longleftrightarrow \quad \frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_r}{(x-a)^r}$$

$$(ax^2+bx+c)^r \longleftrightarrow \frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

where A_i and B_i (with $1 \leq i \leq r$) are constants.

The total number of constants equals the degree of the denominator. These constants can be determined by using the method of undetermined coefficients.

Example 1.6.1:

$$\frac{x+5}{(x-3)(x+1)} = \frac{A}{x-3}+\frac{B}{x+1}.$$

Therefore

$$x+5 = A(x+1)+B(x-3).$$

We could equate coefficients, instead we substitute values chose to make most terms disappear. Substituting $x=-1$ and $x=3$ we obtain

$$4=-4B \qquad \text{and} \qquad 8=4A$$

i.e. $A=2$ and $B=-1$.

Example 1.6.2:

$$\frac{2x^2+x-2}{x^3(x-1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x-1}.$$

Therefore

$$2x^2+x-2 = Ax^2(x-1)+Bx(x-1)+C(x-1)+Dx^3.$$

Substituting $x=0$ and $x=1$ we obtain

$$-2=-C \qquad \text{and} \qquad 1=D$$

Comparing coefficients of the x^3 terms and the x^2 terms we obtain

$$0=A+D \qquad \text{and} \qquad 2=-A+B$$

and hence $A=-1$, $B=1$, $C=2$, $D=1$.

Example 1.6.3:

$$\frac{5x-12}{(x+2)(x^2-2x+3)} = \frac{A}{x+2}+\frac{Bx+C}{x^2-2x+3}$$

as x^2-2x+3 cannot be factorised. Therefore

$$5x-12 = A(x^2-2x+3)+(Bx+C)(x+2).$$

Substituting $x=-2$ we obtain $A=-2$. By comparing coefficients of the x^2 terms and constant terms we obtain $B=2$ and $C=-3$.

Example 1.6.4:

$$\begin{aligned}\frac{3x^3-x^2+2}{x(x^2-1)} &= \frac{3(x^3-x)-x^2+3x+2}{x(x-1)(x+1)}\\ &= 3-\frac{(x^2-3x-2)}{x(x-1)(x+1)}.\end{aligned}$$

Now

$$\frac{(x^2-3x-2)}{x(x-1)(x+1)} = \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}$$

and we can show that $A=2$, $B=-2$ and $C=1$. Therefore

$$\frac{3x^3-x^2+2}{x(x^2-1)} = 3-\frac{2}{x}+\frac{2}{x-1}-\frac{1}{x+1}.$$