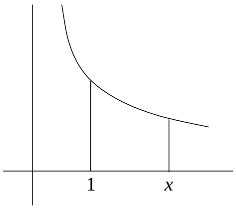
2.4 Logarithm and exponential functions

We first consider the function $f(x) = \ln(x) = \log_e(x)$, the natural logarithm. This is defined for $x > 0$ by

$$\ln(x) = \int_1^x \frac{1}{t}\, dt.$$



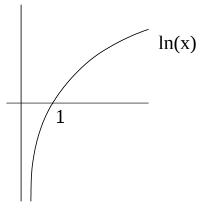
Clearly we have that $\ln(1) = 0$ and, as $\frac{d}{dx}\ln(x) = \frac{1}{x} > 0$, the function is increasing. Therefore $\ln(x)$ is injective:

$$\ln(a) = \ln(b) \quad \text{if and only if} \quad a = b.$$

The function ln satisfies properties similar to those for indices:

$$\begin{aligned} \ln(ab) &= \ln(a) + \ln(b) \\ \ln(a^p) &= p\ln(a) \\ \ln(a^{-1}) &= -\ln(a) \\ \ln(\tfrac{a}{b}) &= \ln(a) - \ln(b) \end{aligned}$$

for all $a, b > 0$ and $p \in \mathbb{R}$.



Example 2.4.1: Find the domain of $\ln(x^2 - 2x - 3)$.
We need $x^2 - 2x - 3 > 0$, i.e. $(x+1)(x-3) > 0$.
Thus either $x < -1$ or $x > 3$.

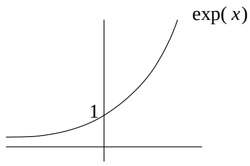
Next we consider the exponential function $f(x) = \exp(x) = e^x$. We set $y = \exp(x)$ if and only if $x = \ln(y)$, so exp is the inverse function to ln. Clearly $\exp(0) = 1$. We define $e = \exp(1)$, so

$$1 = \int_1^e \frac{1}{t}\, dt$$

and $\ln(e) = 1$.

The function exp satisfies

$$\begin{aligned} \exp(a)\exp(b) &= \exp(a+b) \\ \exp(\ln(x)) &= x = \ln(\exp(x)) \\ \exp(-x) &= (\exp(x))^{-1} \end{aligned}$$



Example 2.4.2: If

$$\ln y - \ln(y+3) + \ln 4 = 3x + 2\ln x$$

then find y in terms of x.
Simplifying we obtain

$$\ln\left(\frac{4y}{y+3}\right) = \ln(x^2 e^{3x})$$

and hence

$$\frac{4y}{y+3} = x^2 e^{3x}.$$

Rearranging, we see that

$$4y = x^2 e^{3x}(y+3) \quad \text{and so} \quad y = \frac{3x^2 e^{3x}}{4 - x^2 e^{3x}}.$$

We can also define logarithms to other bases. For $a > 0$ and $y > 0$ set

$$\log_a(y) = x \quad \text{if } y = a^x.$$

Then

$$\begin{aligned} \log_a(a) &= 1 \\ \log_a(xy) &= \log_a x + \log_a y \\ \log_a(x^p) &= p\log_a x \end{aligned}$$

as for natural logarithms.

To change base, suppose that $u = \log_a c$. Then $a^u = c$ and

$$u\log_b a = \log_b c.$$

From this we deduce that

$$\log_a c = \frac{\log_b c}{\log_b a}.$$

In particular, if $b = c$ then

$$\log_a c = \frac{1}{\log_c a}.$$

Example 2.4.3: Solve $2\log_6 x + \log_x 6 = 3$.
First note that for this to be defined we must have $x > 0$.
Using the rules above we have

$$2\log_6 x + \frac{1}{\log_6 x} = 3$$

which becomes

$$2(\log_6 x)^2 - 3\log_6 x + 1 = (2\log_6 x - 1)(\log_6 x - 1) = 0.$$

Thus $\log_6 x = \frac{1}{2}$ or 1, i.e. $x = \sqrt{6}$ or 6.

We have defined $\exp(x)$ as the inverse function to $\ln(x)$, but often denote it by e^x as though it was a power. This is because it is possible to show that

$$\exp(x) = e^x$$

where

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \approx 2.71828.$$

2.5 Solving simultaneous equations

Some sets of equations are too complicated to solve. There may be no exact method for determining solutions, and we may need to use approximate (numerical) solutions. However, here we will concentrate on some simple classes of equations where we can give a procedure for determining the solutions (if any).

First consider the solution of one linear and one quadratic equation.

Example 2.5.1: Solve

$$\begin{aligned} x - y &= 2 \\ 2x^2 - 3y^2 &= 15. \end{aligned}$$

We will reduce the second equation to one involving a single variable by substitution, using the first.

$$2(y+2)^2 - 3y^2 = 15$$

which simplifies to

$$y^2 - 8y + 7 = 0$$

i.e. $y = 1$ or $y = 7$. Therefore the solutions are $y = 1$ and $x = 3$, $y = 7$ and $x = 9$.

We use the method of substitution in many settings. With this as with all methods, we need to be careful that our solutions make sense.

Example 2.5.2: Solve

$$\begin{aligned} x - \sin\theta &= 2 \\ 2x^2 - 3(\sin\theta)^2 &= 15. \end{aligned}$$

Using Example 2.5.1 with $y = \sin\theta$ we see that $\sin\theta = 1$ or $\sin\theta = 7$. But the latter is impossible, and so the only solutions are $\sin\theta = 1$ and $x = 3$, i.e.

$$\theta = \frac{\pi}{2} + 2n\pi \quad (n \in \mathbb{Z}) \quad \text{and} \quad x = 3.$$

Now suppose that we have several equations, each involving several variables, but where all the equations are linear (i.e. involve no products or powers of variables). For example

$$\begin{aligned} 2x + 4y + z &= 7 \\ 3x + 2y + z &= 1. \end{aligned}$$

To solve such equations systematically we use the following procedure. We assume the variables are ordered in some arbitrary way (e.g. x first, then y, then z).

Step 1: Take the first variable, and if necessary reorder the equations so that the first equation contains this variable.

Step 2: Rescale this equation so that the first variable has coefficient 1. Subtract multiples of this equations from the rest to remove all other occurrences of this variable.

Step 3: Take the remaining equations and consider the next variable remaining. Repeat the first two steps for this variable.

Step 4: Repeat Step 3 until no equations, or no variables, remain.

Example 2.5.3: Solve

$$\begin{aligned} 3x + 6y + 6z &= 12 \\ 2x + 4y + 6z &= 6 \\ x + 2y + 4z &= 2. \end{aligned}$$

The first equation involves x, so no need to reorder. Rescaling we obtain

$$\begin{aligned} x + 2y + 2z &= 4 \\ 2x + 4y + 6z &= 6 \\ x + 2y + 4z &= 2. \end{aligned}$$

Subtracting twice the first equation from the second, and the first from the third, we obtain

$$\begin{aligned} x + 2y + 2z &= 4 \\ 2z &= -2 \\ 2z &= -2. \end{aligned}$$

The next remaining variable is z. Consider the last two equations. The first involves z so there is no need to reorder. Rescaling we get

$$\begin{aligned} z &= -1 \\ 2z &= -2 \end{aligned}$$

and subtracting twice the first from the second equation eliminates that equation.

Thus we are left with

$$\begin{aligned} x + 2y + 2z &= 4 \\ z &= -1. \end{aligned}$$

This has general solution $z = -1$ and $x + 2y = 6$. Note that there are many particular solutions, one for each choice of x (or of y).

Once we have reduced our system of equations by the above procedure, solutions are determined by substitution, as in the example. There may be no, one, or many solutions.

Example 2.5.4: Solve

$$\begin{aligned} x \quad +y \quad +z &= 7 \\ x \quad +2y \quad +z &= 4 \\ x \quad +2y \quad +2z &= 5. \end{aligned}$$

This reduces to

$$\begin{aligned} x \quad +y \quad +z &= 7 \\ y \qquad &= -3 \\ y \quad +z &= -2 \end{aligned}$$

and then to

$$\begin{aligned} x \quad +y \quad +z &= 7 \\ y \qquad &= -3 \\ z &= 1. \end{aligned}$$

The unique solution is $z = 1$, $y = -3$, $x = 9$.

Example 2.5.5: Solve

$$\begin{aligned} x \quad +y \quad +z &= 7 \\ x \quad +2y \quad +2z &= 4 \\ 2x \quad +3y \quad +3z &= 5. \end{aligned}$$

This reduces to

$$\begin{aligned} x \quad +y \quad +z &= 7 \\ y \quad +z &= -3 \\ y \quad +z &= -9 \end{aligned}$$

and then to

$$\begin{aligned} x \quad +y \quad +z &= 7 \\ y \quad +z &= -3 \\ 0 &= -6. \end{aligned}$$

This example has no solutions.

While this method may seem complicated in such simple examples, it has the advantage that it works for many equations in many unknowns. Using an ad hoc method, while occasionally quicker, will often lead to confusion.

You will consider this procedure in more detail in the Algebra module, using matrices.

Solving inequalities

If we wish to solve an equation of the form $f(x) > 0$ we usually need to solve $f(x) = 0$ along the way. We also need to be careful if we change the nature of the equation.

Let a, b, and k be real numbers. If $a > b$ then

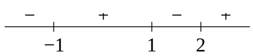
$$\begin{aligned} a \pm k &> b \pm k && \text{for all } k \\ ka &> kb && \text{for all } k > 0 \\ ka &< kb && \text{for all } k < 0. \end{aligned}$$

Example 2.6.1: Solve

$$x^3 - 2x^2 \leq x - 2.$$

First solve $x^3 - 2x^2 - x + 2 = 0$.
Factorising we have

$$(x-1)(x+1)(x-2) = 0.$$



Therefore we must have $x \leq -1$ or $1 \leq x \leq 2$.

Example 2.6.2: Solve

$$\frac{x-2}{x-5} > 3.$$

Method 1:

$$\frac{x-2}{x-5} - 3 > 0 \quad \text{so} \quad \frac{13-2x}{x-5} > 0.$$

Therefore either $13 - 2x > 0$ and $x - 5 > 0$; i.e. $5 < x < \frac{13}{2}$
or $13 - 2x < 0$ and $x - 5 < 0$ which is impossible.

So solution is $5 < x < \frac{13}{2}$.

Method 2: Multiply both sides of the inequality by $(x-5)^2$. We know that this is positive (unless $x = 5$ where the inequality is not defined), so we know how this effects the inequality.

$$(x-5)(x-2) > 3(x-5)^2$$

can be rearranged to

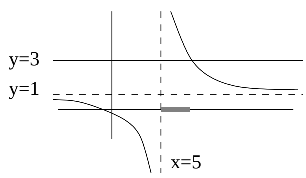
$$(x-5)(x-2-3(x-5)) > 0$$

and so

$$(x-5)(13-2x) > 0.$$

Now solve as in Example 2.6.1.

Method 3: Sketch the curve.



From the graph we can see that the desired solution lies in the shaded region. We now have to find the exact point of intersection (i.e. solve the equality).

Example 2.6.3: Solve

$$\left|\frac{2x-1}{x+2}\right| < 3.$$

Both sides are positive, so squaring each side does not change the inequality.

$$\left(\frac{2x-1}{x+2}\right)^2 < 9.$$

As $(x+2)^2$ is positive whenever the inequality is defined we have

$$(2x-1)^2 < 9(x+2)^2.$$

Simplifying we obtain

$$5x^2 + 40x + 35 > 0 \quad \text{or} \quad (x+1)(x+7) > 0.$$

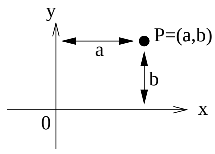
Considering intermediate values we see that the solution is $x < -7$ or $x > -1$.

3. Geometry

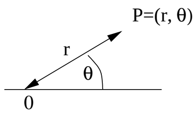
3.1 Coordinate systems

In two dimensions we use two systems of coordinates: Cartesian and polar.

Cartesian coordinates are expressed in terms of orthogonal (i.e. right-angled) axes.



Polar coordinates are expresed in terms of a length and an angle with respect to a fixed axis containing the origin.

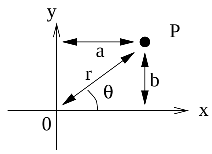


Here $r > 0$ and θ is chosen from a fixed set of representatives of all angles: either $0 \leq \theta < 2\pi$ or $-\pi < \theta \leq \pi$.

The choice of coordinate system depends on the context, as certain curves may be more simply expressed in one form rather than the other.

For example a circle about the origin has polar equation $r = a$.

We can convert between systems.



Polar to Cartesian:

$$x = r\cos\theta \qquad y = r\sin\theta.$$

Cartesian to polar:

$$r = \sqrt{x^2+y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

Example 3.1.1: Find the Cartesian form of the polar equation

$$r = 2A\cos\theta.$$

We have

$$\frac{x}{r} = \cos\theta \quad \text{and} \quad r^2 = x^2 + y^2.$$

Thus the equation becomes

$$r = \frac{2Ax}{r} \quad \text{or} \quad r^2 = 2Ax.$$

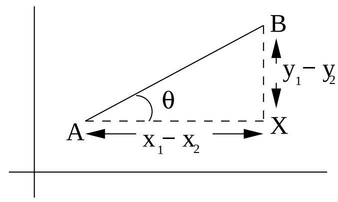
So

$$x^2 + y^2 = 2Ax$$

(which is the equation of a circle).

3.2 Lines

Given two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$, Pythagoras's theorem implies that the distance between A and B is



$$\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}.$$

The midpoint of the line connecting A and B is the point

$$(x_1 + \frac{1}{2}(x_2 - x_1), y_1 + \frac{1}{2}(y_2 - y_1)) = \frac{1}{2}(x_1 + x_2, y_1 + y_2).$$

The gradient of the line joining A and B is defined to be

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \tan\theta$$

where θ is the angle the line makes with the x-axis. This definition does not make sense for vertical lines, which we regard as having infinite gradient.

The equation of our line (if not vertical) is given by

$$y = mx + c$$

where c is the intercept, the value of y at $x = 0$. For vertical lines the equation takes the form

$$x = d.$$

Any line can be written in the form

$$ax + by + c = 0$$

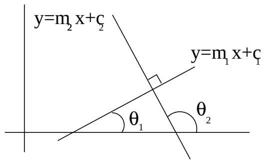
for some choice of a, b, and c.

Given the gradient of a line and a point (a, b) lying on it, the equation of the line is given by

$$y - b = m(x - a)$$

(with the obvious modification for vertical lines).

Now suppose we have two perpendicular (non-vertical) lines.



$m_1 = \tan\theta_1 \quad m_2 = \tan\theta_2$

and $\theta_2 - \theta_1 = \frac{\pi}{2}$.

Then

$$\tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}$$

and we must have

$$1 + \tan\theta_2 \tan\theta_1 = 0$$

i.e. $m_1 m_2 = -1$.

So two lines are perpendicular if and only if $m_1 m_2 = -1$, or one line is horizontal and the other vertical.

Example 3.2.1: Find the equation of the line through $(1, 2)$ and perpendicular to

$$3x - 7y + 2 = 0$$

and find where these lines meet.

Our line is $y - 2 = m(x - 1)$, and the given line is $y = \frac{3x}{7} + \frac{2}{7}$. Thus $\frac{3}{7}m = -1$ and $m = -\frac{7}{3}$. Substituting, we obtain

$$y = -\frac{7}{3}x + \frac{13}{3}$$

or $3y + 7x = 13$. The lines meet when $3y + 7x = 13$ and $3x - 7y = -2$, i.e. at $x = \frac{85}{58}$ and $y = \frac{53}{58}$.

3.3 Circles

The circle of radius r and centre (a, b) has equation

$$(x - a)^2 + (y - b)^2 = r^2.$$

For example, the circle of radius 2 about $(-2, 3)$ has equation

$$(x + 2)^2 + (y - 3)^2 = 4.$$

Expanding, we see that any equation of the form

$$x^2 + y^2 + ex + fy + g = 0$$

for some constants e, f, and g, is a circle.

Example 3.3.1: Find the equation of the tangent to

$$x^2 + y^2 - 4x + 10y - 8 = 0$$

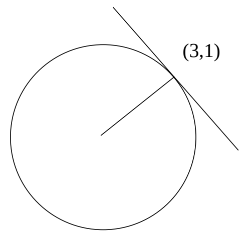
at the point $A = (3, 1)$.

Rearranging, we have the equation

$$(x - 2)^2 + (y + 5)^2 = 37.$$

The centre is at $C = (2, -5)$. The gradient of the line AC is

$$\frac{1 - (-5)}{3 - 2} = 6.$$



The tangent is perpendicular to this, so has gradient $-\frac{1}{6}$, and hence equation

$$(y - 1) = -\frac{1}{6}(x - 3)$$

Example 3.3.2: Find the points of intersection of the circles

$$x^2 + y^2 - 2x - 4y - 20 = 0$$
$$x^2 + y^2 - 32x - 2y + 88 = 0$$

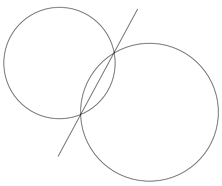
and the equation of the line through these points.

If both equations hold then their difference equals zero:

$$30x - 2y - 108 = 0$$

and so the line of intersection is

$$y = 15x - 54.$$



For the points of intersection, substitute for y in one of the circles.

$$x^2 + (15x - 54)^2 - 2x - 4(15x - 54) - 20 = 0$$

i.e. $(x - 4)(226x - 778) = 0$, so $x = 4$, $y = 6$ or $x = \frac{389}{113}$, $y = -\frac{267}{113}$.