3.4 Conic sections

Circles belong to a special class of curves called conic sections. Other such curves are the ellipse, parabola, and hyperbola.

We will briefly describe the standard conics. These are chosen to have simple equations, and all other conics are variants on them. Our standard conics are all symmetric about the origin and the x-axis. Thus the standard circle is

$$x^2 + y^2 = a^2$$

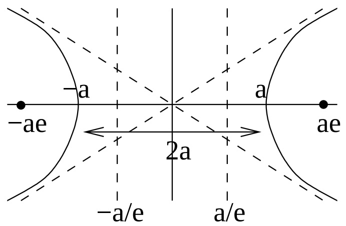
which can be written parametrically as

$$x = a\cos\theta \qquad y = a\sin\theta.$$

There are various ways to define conic sections, for example as the curves arising from different slices through a cone. Each shares the property that:

> The distance of each point on the curve from a fixed point (the focus) and a fixed straight line (the directrix) is a constant ratio e (the eccentricity). (For circles, this has to be interpreted with care.)

The different classes correspond to different ranges of the value of e.



The standard hyperbola is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

with $b^2 = a^2(e^2 - 1)$ for some $e > 1$.

There are two foci, at

$$(ae, 0) \quad (-ae, 0)$$

and two directrices,

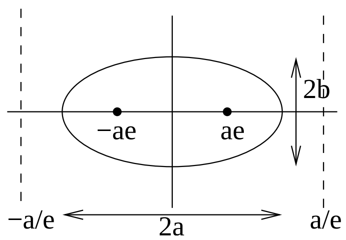
$$x = \frac{a}{e} \quad x = -\frac{a}{e}.$$

We call the shortest distance between the two sections of the curve the major axis, which equals $2a$. This curve has asymptotes

$$y = \frac{bx}{a} \qquad y = -\frac{bx}{a}$$

and parametric equation

$$x = a\sec\theta \qquad y = b\tan\theta.$$



The standard ellipse is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with $b^2 = a^2(1 - e^2)$ for some $0 < e < 1$.

There are two foci, at

$$(ae, 0) \quad (-ae, 0)$$

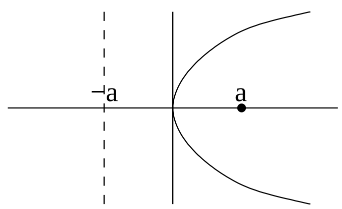
and two directrices,

$$x = \frac{a}{e} \quad x = -\frac{a}{e}.$$

The maximum width of the curve along the x-axis is called the major axis, which equals $2a$, and along the y-axis is called the minor axis, which equals $2b$. Note that the standard ellipse is chosen such that the major axis is longer than the minor axis. This curve has parametric equation

$$x = a\cos\theta \qquad y = b\sin\theta.$$

When $e = 0$ we obtain the case of a circle.



The standard parabola is given by

$$y^2 = 4ax$$

and has eccentricity $e = 1$.

There is one focus, at

$$(a, 0)$$

one directrix,

$$x = -a$$

and an axis at $y = 0$. This curve has parametric equation

$$x = at^2 \qquad y = 2at$$

and the gradient of the curve at $(at^2, 2at)$ is t^{-1}.

We can analyse general conics by using a change of variable to convert them into the standard forms.

Example 3.4.1: Determine the foci and directices of the ellipse

$$\frac{(x-2)^2}{25}+\frac{(y+3)^2}{16}=1.$$

We compare with

$$\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1.$$

To transform in this way we must have

$$X=x-2 \qquad Y=y+3 \qquad a=5 \qquad b=4.$$

Also $b^2=a^2(1-e^2)$ implies that $e=\frac{3}{5}$. Therefore the centre of the ellipse is at $(2,-3)$, the major axis has length $2a=10$ and the minor axis has length $2b=8$.

The foci lie on the major axis at distance $ae=3$ from the centre. So the foci are

$$(5,-3) \qquad (-1,-3).$$

Directrices are perpendicular to the major axis and at distance

$$\frac{a}{e}=\frac{25}{3}$$

from the centre. So the directrices are

$$x=\frac{31}{3} \qquad x=-\frac{19}{3}.$$

Example 3.4.2: An ellipse has foci at $(2,5)$ and $(8,5)$ and eccentricity $e=\frac{1}{4}$. Find its Cartesian equation.

The centre is midway between the foci, so lies at $(5,5)$. The distance from the centre to each focus is $ae=3$, and so $a=12$. Therefore

$$b^2=a^2(1-e^2)=135.$$

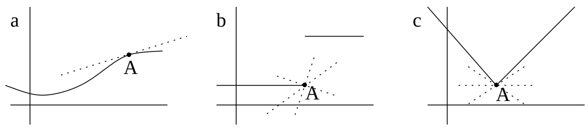
From this we see that the equation is given by

$$\frac{(x-5)^2}{144}+\frac{(y-5)^2}{135}=1.$$

4. Calculus I: Differentiation

4.1 The derivative of a function

Suppose we are given a curve with a point A lying on it. If the curve is 'smooth' at A then we can find a unique tangent to the curve at A:

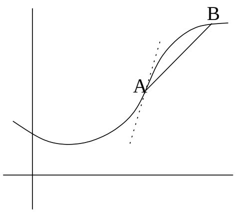


Here the curve in (a) is smooth at A, but the curves in (b) and (c) are not.

If the tangent is unique then the gradient of the curve at A is defined to be the gradient of the tangent to the curve at A.

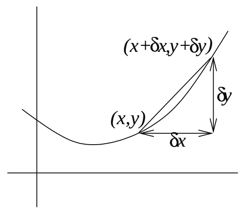
The process of finding the general gradient function for a curve is called differentiation.

Consider the chord AB. As B gets closer to A, the gradient of the chord gets closer to the gradient of the tangent at A.



For $y=f(x)$, the gradient function is defined by

$$\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)=\lim_{\delta x\to 0}\left(\frac{f(x+\delta x)-f(x)}{\delta x}\right).$$



We denote the gradient function by $\frac{dy}{dx}$ or $f'(x)$, and call it the derivative of f. This is not the formal definition of the derivative, as we have not explained exactly what we mean by the limit as $\delta x\to 0$. But this intuitive definition will be sufficient for the basic functions which we consider.

Example 4.1.1: Take $f(x)=c$, a constant function.

At every x the gradient is 0, so $f'(x)=0$ for all x.
Or

$$\frac{f(x+\delta x)-f(x)}{\delta x}=\frac{c-c}{\delta x}=0.$$

Example 4.1.2: Take $f(x)=ax$.

At every x the gradient is a, so $f'(x)=a$ for all x.
Or

$$\frac{f(x+\delta x)-f(x)}{\delta x}=\frac{a(x+\delta x)-ax}{\delta x}=\frac{a\delta x}{\delta x}=a.$$

Example 4.1.3: Take $f(x)=x^2$.

Now we need to consider the second formulation, as we cannot simply read the gradient off from the graph.

$$\begin{aligned}\frac{f(x+\delta x)-f(x)}{\delta x} &= \frac{(x+\delta x)^2-x^2}{\delta x}\\ &= \frac{x^2+2x\delta x+(\delta x)^2-x^2}{\delta x}\\ &= \frac{\delta x(2x+\delta x)}{\delta x}=2x+\delta x.\end{aligned}$$

The limit as δx tends to 0 is $2x$, so $f'(x)=2x$.

Example 4.1.4: Take $f(x) = \frac{1}{x}$.

$$\begin{aligned}\frac{f(x+\delta x)-f(x)}{\delta x} &= \frac{1}{\delta x}\left(\frac{1}{x+\delta x}-\frac{1}{x}\right)\\ &= \frac{x-(x+\delta x)}{(\delta x)(x+\delta x)x}\\ &= \frac{-\delta x}{(\delta x)(x+\delta x)x} = \frac{-1}{(x+\delta x)x}.\end{aligned}$$

The limit as δx tends to 0 is $-\frac{1}{x^2}$, so $f'(x) = -\frac{1}{x^2}$.

Example 4.1.5: Take $f(x) = x^n$ with $n \in \mathbb{N}$ and $n > 1$.

Recall that

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1})$$

and so

$$\frac{a^n - b^n}{a-b} = a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1}$$

where the sum has n terms. As $a \to b$ we have

$$\lim_{a\to b}\left(\frac{a^n-b^n}{a-b}\right) = \lim_{a\to b}(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1}) = nb^{n-1}.$$

If $a = x + \delta x$ and $b = x$ then

$$\lim_{\delta x\to 0}\left(\frac{f(x+\delta x)-f(x)}{\delta x}\right) = \lim_{a\to b}\left(\frac{a^n-b^n}{a-b}\right) = nb^{n-1} = nx^{n-1}.$$

Hence $f'(x) = nx^{n-1}$.

Example 4.1.6: $f(x) = \sin x$.

We use the identity for $\sin A + \sin B$.

$$f(x+\delta x) - f(x) = 2\sin\left(\frac{\delta x}{2}\right)\cos\left(x + \frac{\delta x}{2}\right)$$

and so

$$\frac{f(x+\delta x)-f(x)}{\delta x} = \frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}\cos\left(x+\frac{\delta x}{2}\right).$$

We need the following fact (which we will not prove here):

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

and so

$$f'(x) = \lim_{\delta x\to 0}\frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}\cos\left(x+\frac{\delta x}{2}\right) = \cos(x).$$

Some standard derivatives, which must be memorised:

$f(x)$	$f'(x)$
x^k	kx^{k-1}
e^x	e^x
$\ln x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$

Some of these results can be derived from the results in the following sections, or from first principles. However it is much more efficient to know them.

4.2 Differentiation of compound functions

Once we know a few basic derivatives, we can determine many others using the following rules:
Let $u(x)$ and $v(x)$ be functions of x, and a and b be constants.

	Function	Derivative	
Sum and difference	$au \pm bv$	$a\frac{du}{dx} \pm b\frac{dv}{dx}$	
Product	uv	$v\frac{du}{dx} + u\frac{dv}{dx}$	
Quotient	$\frac{u}{v}$	$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	
Composite	$u(v(x))$	$\frac{du}{dz}\cdot\frac{dz}{dx}$	where $z = v(x)$.

The final rule above is known as the chain rule and has the following special case

$$u(ax+b) \qquad a\frac{du}{dx}(ax+b)$$

For example, the derivative of $\sin(ax+b)$ is $a\cos(ax+b)$.

Example 4.2.1: Differentiate

$$y = 2x^5 - 3x^3 + \frac{4}{x^2}.$$

$$\frac{dy}{dx} = 10x^4 - 9x^2 - \frac{8}{x^3}.$$

Example 4.2.2: Differentiate

$$y = \frac{x^2-1}{x^2+1}.$$

$$\frac{dy}{dx} = \frac{(x^2+1)2x - (x^2-1)2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}.$$

Example 4.2.3: Differentiate

$$y = x^2\ln(x+3).$$

$$\frac{dy}{dx} = 2x\ln(x+3) + \frac{x^2}{x+3}.$$

Example 4.2.4: Differentiate $y = e^{5x}$.

Set $z = 5x$, then

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = e^z 5 = 5e^{5x}.$$

Example 4.2.5: Differentiate $y = 4\sin(2x+3)$.

Set $z = 2x + 3$, then

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = 4\cos(z)2 = 8\cos(2x+3).$$

As we have already noted, some of the standard derivatives can be deduced from the others.

Example 4.2.6: Differentiate

$$y = \tan x = \frac{\sin x}{\cos x}.$$

$$\frac{dy}{dx} = \frac{\cos x\cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

Example 4.2.7: $y = \operatorname{cosec} x = \frac{1}{\sin x}$.

$$\frac{dy}{dx} = \frac{\sin x.(0) - 1.\cos x}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = -\operatorname{cosec} x \cot x.$$

Example 4.2.8: $y = \ln(x + \sqrt{x^2+1})$, i.e. $y = \ln u$ where $u = x + \sqrt{x^2+1}$.

$$\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} \quad \text{and} \quad \frac{du}{dx} = 1 + \frac{(x^2+1)^{-\frac{1}{2}}}{2}.2x$$

so

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$$
$$= \frac{1}{x+\sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right) = \frac{1}{\sqrt{x^2+1}}.$$

Example 4.2.9: $y = x^x$.

We have $y = (e^{\ln x})^x = e^{(x \ln x)}$, i.e. $y = e^u$ where $u = x \ln x$.

$$\frac{dy}{dx} = e^u \frac{du}{dx} = e^{x \ln x}(\ln(x) + 1) = x^x(\ln(x)+1).$$

4.3 Higher derivatives

The derivative $\frac{dy}{dx}$ is itself a function, so we can consider its derivative. If $y = f(x)$ then we denote the second derivative, i.e. the derivative of $\frac{dy}{dx}$ with respect to x, by $\frac{d^2y}{dx^2}$ or $f''(x)$. We can also calculate the higher derivatives $\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$.

Example 4.3.1: $y = \ln(1+x^2)$.

Let $z = \frac{dy}{dx} = \frac{2x}{1+x^2}$.

$$\frac{d^2y}{dx^2} = \frac{dz}{dx} = \frac{(1+x^2).2 - 2x(2x)}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2}.$$

Example 4.3.2: Show that $y = e^{-x}\sin(2x)$ satisfies

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0.$$

$$\frac{dy}{dx} = -e^{-x}\sin 2x + 2e^{-x}\cos 2x = e^{-x}(2\cos 2x - \sin 2x)$$

$$\frac{d^2y}{dx^2} = -e^{-x}(2\cos 2x - \sin 2x) + e^{-x}(-4\sin 2x - 2\cos 2x)$$
$$= e^{-x}(-3\sin 2x - 4\cos 2x).$$

Writing s for $\sin 2x$ and c for $\cos 2x$ we have

$$y'' + 2y' + 5y = e^{-x}(-3s - 4c - 2s + 4c + 5s) = 0.$$

Example 4.3.3: Evaluate

$$\frac{d^3}{dx^3}\left(\frac{1+3x^2}{(1+x)^2(1+3x)}\right)$$

at $x = 0$.

We could use the quotient rule, but this will get complicated. Instead we use partial fractions.

$$y = \frac{1+3x^2}{(1+x)^2(1+3x)} = \frac{A}{1+x} + \frac{B}{(1+x)^2} + \frac{C}{1+3x}.$$

We obtain (check!) $A = 0$, $B = -2$, and $C = 3$.

Now

$$\frac{dy}{dx} = \frac{4}{(1+x)^3} - \frac{9}{(1+3x)^2}$$
$$\frac{d^2y}{dx^2} = \frac{-12}{(1+x)^4} + \frac{54}{(1+3x)^3}$$
$$\frac{d^3y}{dx^3} = \frac{48}{(1+x)^5} - \frac{54 \times 9}{(1+3x)^4}$$

and substituting $x = 0$ we obtain that

$$\frac{d^3y}{dx^3}(0) = 48 - 486 = -438.$$

Generally it is hard to give a simple formula for the nth derivative of a function. However, in some cases it is possible. The following can be proved by induction.

Example 4.3.4: $y = e^{ax}$.

$$\frac{dy}{dx} = ae^{ax} \quad \text{and} \quad \frac{d^2y}{dx^2} = a^2e^{ax}.$$

We can show that

$$\frac{d^ny}{dx^n} = a^ne^{ax}.$$

Example 4.3.5: $y = \sin(ax)$.

$$\begin{aligned} y' &= a\cos(ax) &&= a\sin(ax + \tfrac{\pi}{2}) \\ y'' &= -a^2\sin(ax) &&= a^2\sin(ax+\pi) \\ y''' &= -a^3\cos(ax) &&= a^3\sin(ax + \tfrac{3\pi}{2}) \\ y^{(iv)} &= a^4\sin(ax) &&= a^4\sin(ax+2\pi). \end{aligned}$$

We can show that

$$\frac{d^ny}{dx^n} = a^n\sin(ax + \frac{n\pi}{2}).$$

4.4 Differentiating implicit functions

Sometimes we cannot rearrange a function into the form $y = f(x)$, or we may wish to consider the original form anyway (for example, because it is simpler). However, we may still wish to differentiate with respect to x.

Given a function $g(y)$ we have from the chain rule

$$\frac{d}{dx}(g(y)) = \frac{d}{dy}(g(y))\frac{dy}{dx}.$$

Example 4.4.1: $x^2 + 3xy^2 - y^4 = 2$.

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + 3xy^2 - y^4) = \frac{\mathrm{d}}{\mathrm{d}x}(2) = 0.$$

Therefore we have

$$\begin{aligned} 2x + \frac{\mathrm{d}}{\mathrm{d}x}(3xy^2) - \frac{\mathrm{d}}{\mathrm{d}x}(y^4) &= 0 \\ 2x + 3y^2 + 3x\frac{\mathrm{d}}{\mathrm{d}x}(y^2) - 4y^3\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ 2x + 3y^2 + 6xy\frac{\mathrm{d}y}{\mathrm{d}x} - 4y^3\frac{\mathrm{d}y}{\mathrm{d}x} &= 0. \end{aligned}$$

Example 4.4.2: $\frac{2}{x^2} + \frac{3}{y^2} = \frac{1}{2}$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2}{x^2} + \frac{3}{y^2}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\right) = 0.$$

Therefore we have

$$\begin{aligned} -\frac{4}{x^3} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{3}{y^2}\right) &= 0 \\ -\frac{4}{x^3} - \frac{6}{y^3}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0. \end{aligned}$$

4.5 Differentiating parametric equations

Sometimes there is no easy way to express the relationship between x and y directly in a single equation. In such cases it may be possible to express the relationship between them by writing each in terms of a third variable. We call such equations parametric equations as both x and y depend on a common parameter.

Example 4.5.1: $x = t^3 \qquad y = t^2 - 4t + 2$.

Although we can write this in the form

$$y = x^{\frac{2}{3}} - 4x^{\frac{1}{3}} + 2$$

the parametric version is easier to work with.

To differentiate a parametric equation in the variable t we use

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} \quad \text{and} \quad \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Example 4.5.1: (Continued.)

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2t - 4 \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2t-4}{3t^2}.$$

Example 4.5.2: Find the second derivative with respect to x of

$$x = \sin\theta \qquad y = \cos 2\theta.$$

We have

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \cos\theta \qquad \frac{\mathrm{d}y}{\mathrm{d}\theta} = -2\sin 2\theta.$$

Therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2\sin 2\theta}{\cos\theta} = -4\sin\theta.$$

Now

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}(-4\sin\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}(-4\sin\theta)\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{-4\cos\theta}{\cos\theta} = -4.$$

Note: The rules so far may suggest that derivatives can be treated just like fractions. However

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \neq \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\frac{\mathrm{d}^2t}{\mathrm{d}x^2}$$

in general. Moreover

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \neq \left(\frac{\mathrm{d}^2x}{\mathrm{d}y^2}\right)^{-1}.$$

Example 4.5.2: (Continued.) We have

$$\frac{\mathrm{d}^2y}{\mathrm{d}\theta^2} = -4\sin 2\theta = -8\sin\theta\cos\theta$$

and

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}\theta}(\sec\theta)\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right) = \sec^2\theta\tan\theta.$$

Therefore

$$\frac{\mathrm{d}^2y}{\mathrm{d}\theta^2}\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} = -8\sin\theta\cos\theta\sec^2\theta\tan\theta = -8\tan^2\theta \neq -4 = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}.$$