4.6 Tangents and normals to curves

We have already defined the value of the derivative f' of a function f at a point x_0 to be the gradient of f at x_0. Thus we can easily use the derivative to write down the equation of the tangent to that point. Using the equation for a line passing through $(x_0, f(x_0))$ we have that the tangent to f at x_0 is

$$y - f(x_0) = \frac{\mathrm{d}y}{\mathrm{d}x}(x_0)(x - x_0).$$

The normal to f at x_0 is the line passing through $(x_0, f(x_0))$ perpendicular to the tangent. This has equation

$$y - f(x_0) = \frac{-1}{\frac{\mathrm{d}y}{\mathrm{d}x}(x_0)}(x - x_0)$$

(when this makes sense).

Example 4.6.1: Find the equation of the tangent and normal to the curve

$$y = x^2 - 6x + 5$$

at the point $(2, -3)$.

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 6$$

and hence $\frac{\mathrm{d}y}{\mathrm{d}x}(2) = 4 - 6 = -2$. Hence the equation of the tangent is

$$y + 3 = -2(x - 2) \quad \text{i.e.} \quad y = -2x + 1.$$

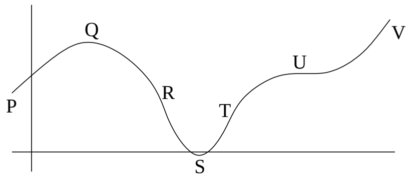
The gradient of the normal is $\frac{-1}{-2} = \frac{1}{2}$, and hence the equation of the normal is

$$y + 3 = \frac{1}{2}(x - 2) \quad \text{i.e.} \quad y = \frac{x}{2} - 4.$$

4.7 Stationary points and points of inflexion

We can tell a lot about a function from its derivatives.

Example 4.7.1:



If $f'(x) > 0$ for $a < x < b$ then f is increasing on $a < x < b$

e.g. arcs PQ, SU, UV.

If $f'(x) < 0$ for $a < x < b$ then f is decreasing on $a < x < b$

e.g. arc QS.

A stationary point on a curve $y = f(x)$ is a point $(x_0, f(x_0))$ such that $f'(x_0) = 0$. These come in various forms:

Type	Test $f'(x)$	Test $f''(x)$
Local maximum	Changes from $+$ to $-$	$-$ve
Local minimum	Changes from $-$ to $+$	$+$ve
Point of inflexion	No sign change	(see below)

e.g. Q is a max, S is a min, U is a point of inflexion.

A point of inflexion is one where $f''(x_0) = 0$ and f'' changes sign at x_0.

e.g. R, T, U.

If $f''(x) > 0$ for $a < x < b$ then f is concave up on $a < x < b$

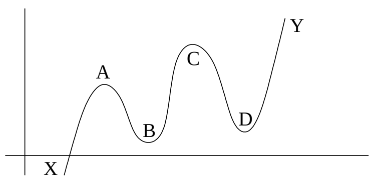
e.g. arc RST.

If $f''(x) < 0$ for $a < x < b$ then f is concave down on $a < x < b$

e.g. arc PQR.

Note that the maxima and minima above are only local. This means that in a small region about the given point they are extremal values, but perhaps not over the whole curve. Extremal values for the whole curve are called global maxima or minima.

Example 4.7.2: Consider the function f on the domain $X \leq x \leq Y$ given by the graph



Both A and C are local maxima, and B and D are local minima. However the global maximum is at Y and the global minimum at X.

Example 4.7.3: Find the stationary values and points of inflexion of

$$y = 3x^4 + 8x^3 - 6x^2 - 24x + 2.$$

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 12x^3 + 24x^2 - 12x - 24$$

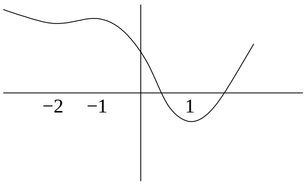
and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 36x^2 + 48x - 12.$$

Stationary points when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, i.e. (check) $x = 1, -1, -2$.

	y'	y''	
$(1,-17)$	$-0+$	72	Min
$(-1,15)$	$+0-$	-24	Max
$(-2,10)$	$-0+$	36	Min

Points of inflexion at $x = \frac{1}{3}(-2 \pm \sqrt{7})$, i.e. $(x, y) \approx (0.22, -3.36)$ and $(x, y) \approx (-1.55, 12.32)$.



Example 4.7.4: Find the stationary points of the curve

$$f(x) = 6 \ln\left(\frac{x}{7}\right) + (x-1)(x-7).$$

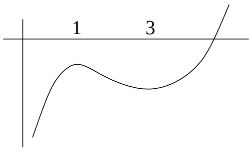
Deduce that $f(x) = 0$ has only one solution, and state its value.

$$\frac{dy}{dx} = \frac{6}{x} + 2x - 8 \qquad \frac{d^2y}{dx^2} = -\frac{6}{x^2} + 2.$$

We have $f'(x) = 0$ when $2x^2 - 8x + 6 = 0$, i.e. $x = 1$ or 3.

$$f''(1) = -4 \text{ so there is a local max at } (1, -6\ln 7).$$
$$f''(3) = \tfrac{4}{3} \text{ so there is a local min at } (3, -6\ln(\tfrac{7}{3}) - 8).$$

For large x the function f is large and positive. Therefore the curve is of the form



It cannot cross the x-axis again as there are no other turning points, so $f(x) = 0$ has only one solution. By inspection, $x = 7$ is a root.

Example 4.7.5: Find the least value of

$$y = a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x$$

where a and b are positive constants and $0 < x < \frac{\pi}{2}$.

$$\begin{aligned}
\tfrac{dy}{dx} &= 2a^2 \sec x(\sec x \tan x) + 2b^2 \operatorname{cosec} x(-\operatorname{cosec} x \cot x)\\
&= 2a^2 \sec^2 x \tan x - 2b^2 \operatorname{cosec}^2 x \cot x\\
&= 2a^2 \frac{\sin x}{\cos^3 x} - 2b^2 \frac{\cos x}{\sin^3 x}\\
&= \frac{2a^2 \sin^4 x - 2b^2 \cos^4 x}{\cos^3 x \sin^3 x}.
\end{aligned}$$

Stationary point are where $y' = 0$, i.e. where

$$2a^2 \sin^4 x - 2b^2 \cos^4 x = 0.$$

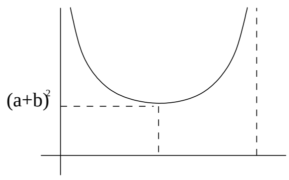
This can be rearranged to give

$$\tan^4 x = \frac{b^2}{a^2} \quad \text{or} \quad \tan^2 x = \frac{b}{a}.$$

Since $0 < x < \frac{\pi}{2}$ we have $\tan x > 0$, and so $\tan x = \sqrt{b/a}$, and there is precisely one stationary point.

Since $y \to \infty$ as $x \to 0$ or $x \to \frac{\pi}{2}$, the stationary point must be a minimum. Substituting for $\tan x$ in y gives

$$\begin{aligned}
y &= a^2(1 + \tan^2 x) + b^2(1 + \cot^2 x)\\
&= a^2\left(1 + \frac{b}{a}\right) + b^2\left(1 + \frac{a}{b}\right)\\
&= a^2 + 2ab + b^2 = (a+b)^2
\end{aligned}$$



5. Calculus II: Integration

5.1 Basic theory

We will define the integral of a function $f(x)$ to be its antiderivative:

$$\int f(x)\,dx = F(x) + C$$

where C is a constant and $F(x)$ is a function with $\frac{dF}{dx} = f(x)$. Any two functions F and G with $\frac{dF}{dx} = \frac{dG}{dx} = f(x)$ must satisfy $\frac{d}{dx}(F - G) = 0$, i.e. $F - G$ is some constant function. Thus the integral is only defined up to the undetermined constant C.

From our standard results for differentiation we deduce the following integrals, which must be memorised.

$f(x)$	$\int f(x)\,dx$
x^k $(k \neq -1)$	$\frac{1}{k+1}x^{k+1} + C$
x^{-1}	$\ln x + C$
e^x	$e^x + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\tan x$	$-\ln(\cos x) + C$

There are obvious extensions of these results, replacing x by $ax+b$. For example, for $k \neq -1$ we have

$$\int (ax+b)^k \, dx = \frac{(ax+b)^{k+1}}{a(k+1)} + C$$

and

$$\int \sin(ax+b) \, dx = \frac{-\cos(ax+b)}{a} + C.$$

etc. We also have for functions f and g and constants a and b that

$$\int af + bg \, dx = a \int f \, dx + b \int g \, dx.$$

Example 5.1.1:

$$\int x^7 + \frac{3}{x^2} - \sqrt{x} dx = \int x^7 dx + 3 \int x^{-2} dx - \int x^{\frac{1}{2}} dx$$
$$= \frac{x^8}{8} - \frac{3}{x} - \frac{2}{3} x^{\frac{3}{2}} + C.$$

Example 5.1.2:

$$\int \frac{1}{(2x+3)^4} dx = \frac{(2x+3)^{-3}}{(-3).2} + C = \frac{-1}{6(2x+3)^3} + C.$$

For more complicated rational functions we usually simplify first using partial fractions.

Example 5.1.3:

$$\int \frac{1}{(x-1)(x-2)} dx = \int \frac{-1}{(x-1)} + \frac{1}{x-2} dx$$
$$= -\ln(x-1) + \ln(x-2) + C = \ln\left(\frac{x-2}{x-1}\right) + C.$$

Example 5.1.4:

$$\int \frac{1+3x^2}{(1+x)^2(1+3x)} dx = \int \frac{-2}{(1+x)^2} + \frac{3}{1+3x} dx = \frac{2}{1+x} + \ln(1+3x) + C.$$

Example 5.1.5:

$$\int \sin 5x \, dx = -\frac{1}{5} \cos 5x + C.$$

For more complicated integrals involving trigonometric functions, we typically use standard identities to simplify the integral.

Example 5.1.6:

$$\int \sin^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x) dx = \frac{x}{2} - \frac{1}{4} \sin 2x + C.$$

Example 5.1.7:

$$\int \sin 3x \cos x dx = \int \frac{\sin(3x+x) + \sin(3x-x)}{2} dx$$
$$= \int \frac{1}{2}(\sin 4x + \sin 2x) dx = -\frac{1}{8} \cos 4x - \frac{1}{4} \cos 2x + C.$$

Sometimes it is not so easy to spot the integral of a function.

Example 5.1.8: $\int 2xe^{x^2} dx.$

This does not correspond to one of our standard integrals. However, by inspection we can observe that

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^{x^2}) = 2xe^{x^2}$$

using the chain rule, and hence

$$\int 2xe^{x^2} dx = e^{x^2} + C.$$

We would like to formalise this procedure.

5.2 Method of substitution

Recall the chain rule for differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = f'(g(x))g'(x).$$

Integrating both sides we obtain

$$\int f'(g(x))g'(x) \, dx = f(g(x)) + C.$$

Writing $u = g(x)$ this becomes

$$\int f'(u) \frac{\mathrm{d}u}{\mathrm{d}x} \, dx = f(u) + C$$

and so we have

$$\int f'(g(x))g'(x) \, dx = \int f'(u) \, du$$

where $u = g(x)$.

Example 5.2.1: We return to example 5.1.8, and recalculate
$$\int 2xe^{x^2}\,dx.$$
Let $u = x^2$, so $\frac{du}{dx} = 2x$. Then
$$\int 2xe^{x^2}\,dx = \int e^u \frac{\mathrm{d}u}{\mathrm{d}x}\,dx = \int e^u\,du = e^u + C = e^{x^2} + C.$$

Example 5.2.2: Integrate
$$\int x^2(x^3+1)^{\frac{3}{2}}\,dx.$$
Let $u = x^3+1$, so $\frac{du}{dx} = 3x^2$. Then
$$\begin{aligned}\int x^2(x^3+1)^{\frac{3}{2}}\,dx &= \int \frac{u^{\frac{3}{2}}}{3}\frac{\mathrm{d}u}{\mathrm{d}x}\,dx\\ &= \int \frac{u^{\frac{3}{2}}}{3}\,du = \frac{2}{15}u^{\frac{5}{2}} + C = \frac{2}{15}(x^3+1)^{\frac{5}{2}} + C.\end{aligned}$$

Example 5.2.3: Integrate
$$\int \sin^4 x \cos x\,dx.$$
Let $u = \sin x$, so $\frac{du}{dx} = \cos x$. Then
$$\int \sin^4 x\cos x\,dx = \int u^4\,du = \frac{u^5}{5} + C = \frac{\sin^5 x}{5} + C.$$

Example 5.2.4: Integrate
$$\int \tan x\,dx.$$
First note that
$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx.$$
Let $u = \cos x$, so $\frac{du}{dx} = -\sin x$. Then
$$\int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{u}\,du = -\ln(u)+C = -\ln(\cos x)+C = \ln(\sec x)+C.$$

5.3 Inverse substitution

In the last section we substituted
$$\begin{aligned} f'(g(x)) &\longrightarrow f'(u)\\ g'(x)\,dx &\longrightarrow du.\end{aligned}$$

Next we consider the inverse substitution. Replacing f' by h and interchanging the roles of x and u we have
$$\int h(g(u))g'(u)\,du = \int h(x)\,dx$$
where $x = g(u)$. Therefore we can substitute
$$\begin{aligned} h(x) &\longrightarrow h(g(u))\\ dx &\longrightarrow g'(u)\,du = \frac{\mathrm{d}x}{\mathrm{d}u}\,du.\end{aligned}$$

Example 5.3.1: Integrate
$$\int \frac{1}{1+\sqrt{x}}\,dx.$$
Let $\sqrt{x} = u$, so $x = u^2$ and $\frac{dx}{du} = 2u$. Then
$$\begin{aligned}\int \frac{1}{1+\sqrt{x}}\,dx &= \int \frac{1}{1+u}2u\,du\\ &= \int 2 - \frac{2}{1+u}\,du\\ &= 2u - 2\ln(1+u) + C = 2\sqrt{x} - 2\ln(1+\sqrt{x}) + C.\end{aligned}$$

Example 5.3.2: Integrate
$$\int \frac{x-2}{\sqrt{2x+3}}\,dx.$$
Let $u = \sqrt{2x+3}$, so $2x+3 = u^2$ and $\frac{dx}{du} = u$. Then
$$\begin{aligned}\int \frac{x-2}{\sqrt{2x+3}}\,dx &= \int \frac{\frac{1}{2}(u^2-3)-2}{u}u\,du\\ &= \int \frac{1}{2}(u^2-7)\,du\\ &= \frac{u^3}{6} - \frac{7u}{2} + C = \frac{u}{6}(u^2-21) + C\\ &= \frac{\sqrt{2x+3}}{6}(2x-18) + C.\end{aligned}$$

Example 5.3.3: Integrate
$$\int \frac{1}{(4-x^2)^{\frac{3}{2}}}\,dx.$$
Let $x = 2\sin\theta$, so $\frac{dx}{d\theta} = 2\cos\theta$, and $4 - x^2 = 4\cos^2\theta$. Then
$$\begin{aligned}\int \frac{1}{(4-x^2)^{\frac{3}{2}}}\,dx &= \int \frac{2\cos\theta}{8\cos^3\theta}\,d\theta\\ &= \frac{1}{4}\int \sec^2\theta\,d\theta = \frac{1}{4}\tan\theta + C\\ &= \frac{1}{4}\frac{\sin\theta}{\sqrt{1-\sin^2\theta}} + C = \frac{1}{4}\frac{x}{\sqrt{4-x^2}} + C.\end{aligned}$$

5.4 Integration by parts

Recall the rule for differentiating a product of functions:
$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = \frac{\mathrm{d}u}{\mathrm{d}x}.v + u.\frac{\mathrm{d}v}{\mathrm{d}x}.$$
Using the antiderivative this becomes
$$uv = \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,dx + \int u\frac{\mathrm{d}v}{\mathrm{d}x}\,dx.$$
Therefore
$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,dx = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,dx.$$

Example 5.4.1: Calculate

$$\int x \cos x \, dx.$$

Let $u = x$ and $\frac{dv}{dx} = \cos x$. Then $\frac{du}{dx} = 1$ and $v = \sin x$.

$$\begin{aligned} \int x \cos x \, dx &= x \sin x - \int (\sin x).1 \, dx \\ &= x \sin x + \cos x + C. \end{aligned}$$

Example 5.4.2: Calculate

$$S = \int x^2 e^{3x} \, dx.$$

Let $u = x^2$ and $\frac{dv}{dx} = e^{3x}$. Then $\frac{du}{dx} = 2x$ and $v = \frac{1}{3}e^{3x}$.

$$S = \frac{x^2}{3}e^{3x} - \int \frac{2x}{3}e^{3x} \, dx = \frac{x^2}{3}e^{3x} - T.$$

Now use integration by parts again to determine T

Let $u = \frac{2x}{3}$ and $\frac{dv}{dx} = e^{3x}$. Then $\frac{du}{dx} = \frac{2}{3}$ and $v = \frac{1}{3}e^{3x}$.

$$\begin{aligned} T &= \frac{2x}{3}\frac{e^{3x}}{3} - \int \frac{2}{9}e^{3x} \, dx \\ &= \frac{2x}{9}e^{3x} - \frac{2}{27}e^{3x} + C. \end{aligned}$$

So

$$S = \left(\frac{x^2}{3} - \frac{2x}{9} + \frac{2}{27}\right)e^{3x} + C.$$

Using this method we can integrate another of our standard functions.

Example 5.4.3: Calculate

$$\int \ln(x) \, dx.$$

Let $u = \ln(x)$ and $\frac{dv}{dx} = 1$. Then $\frac{du}{dx} = \frac{1}{x}$ and $v = x$.

$$\begin{aligned} \int \ln(x) \, dx &= x \ln(x) - \int \tfrac{x}{x} \, dx \\ &= x \ln(x) - x + C, \end{aligned}$$

Next time we will see how integration by parts can be used in more complicated examples.