We saw in Example 5.4.2 that we sometimes need to apply integration by parts several times in the course of a single calculation.

**Example 5.4.4:** For \( n \geq 0 \) let

\[
S_n = \int x^n \cos 2x \, dx.
\]

Find an expression for \( S_n \) in terms of \( S_{n-2} \) and hence evaluate \( S_4 \).

Let \( u = x^n \) and \( \frac{dv}{dx} = \cos 2x \). Then \( \frac{du}{dx} = nx^{n-1} \) and \( v = \frac{1}{2} \sin(2x) \).

Integrating by parts we have

\[
\begin{align*}
S_n &= x^n \sin 2x \bigg|_0^S - \int_0^S nx^{n-1} \sin 2x \, dx \\
&= \frac{x^n}{2} \sin(2x) - \frac{n}{2} x^{n-1} \sin 2x \\
&= \frac{x^n}{2} \sin(2x) + \frac{n}{4} x^{n-1} \cos 2x \\
&= \frac{n(n-1)}{4} x^{n-2} \cos 2x \\
&= \frac{x^n}{2} \sin(2x) + \frac{n}{4} x^{n-1} \cos 2x - \frac{n(n-1)}{4} S_{n-2}
\end{align*}
\]

where the second equality follows using integration by parts with \( u = \frac{x^n}{2} \) and \( \frac{dv}{dx} = \sin 2x \). Thus we have found a formula for \( S_n \) in terms of \( S_{n-2} \).

Clearly \( S_0 = \int \cos 2x \, dx = \frac{1}{2} \sin 2x + C \). Hence

\[
S_2 = \frac{x^2}{2} \sin(2x) + \frac{2}{4} x^2 \cos 2x - \frac{1}{4} \sin 2x + C
\]

for some constant \( C \) and

\[
S_4 = \frac{x^4}{2} \sin(2x) + \frac{4}{2} x^4 \cos 2x \\
- 3 \left( \frac{x^2}{2} \sin(2x) + \frac{1}{2} x^3 \cos 2x - \frac{1}{4} \sin 2x + C \right) \\
= \frac{1}{2} (2x^4 - 6x^2 + 3) \sin 2x + \frac{1}{2} (2x^3 - 3x) \cos 2x + C
\]

for some constant \( C \).

Integrating by parts again we have

\[
\int e^x \cos x \, dx = e^x \sin x - \left[ -e^x \cos x + \int e^x \cos x \, dx \right].
\]

The final integral is identical to that we first wished to calculate, however we can now rearrange this formula to obtain

\[
2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x
\]

from which we deduce that

\[
\int e^x \cos x \, dx = \frac{1}{2} (e^x \sin x + e^x \cos x)
\]

5.5 The definite integral

If

\[
\int_a^b g(x) \, dx = G(b) - G(a)
\]

then we define

\[
\left[ G(x) \right]_a^b = G(b) - G(a)
\]

which we also denote by

\[
\begin{align*}
\left[ G(x) \right]_a^b &= G(b) - G(a) \\
&= \frac{1}{5} - 0 = \frac{1}{5}
\end{align*}
\]

In the next example we will apply Example 5.2.3.

**Example 5.5.2:**

\[
\int_0^\frac{\pi}{2} \sin^2 x \cos x \, dx = \left[ \frac{1}{3} \sin^3 x \right]_0^\frac{\pi}{2}
\]

\[
= \frac{1}{3} \left( \frac{1}{2} - 0 \right) = \frac{1}{6}
\]
When integrating a definite integral by substitution we must be careful to convert the limits into the new variable.

**Example 5.5.3:** Calculate
\[ \int_0^2 \sqrt{4 - x^2} \, dx. \]

Let \( x = 2 \sin \theta \), so \( \frac{dx}{d\theta} = 2 \cos \theta \). We have
\[ 4 - x^2 = 4 - 4 \sin^2 \theta = 4 \cos^2 \theta \]
and in changing variable we have
\[
\begin{align*}
  x = 0 & \quad \Rightarrow \quad \theta = 0 \\
  x = 2 & \quad \Rightarrow \quad \theta = \frac{\pi}{2}
\end{align*}
\]

This follows from the definition of integration as a measure of content:
\[
\text{area} = \sum_{k=1}^{n} f(x_k) \delta x_k
\]

and the fundamental theorem of calculus which states that this definition agrees with that coming from the antiderivative.

Note that this result relies on the convention that area below the x-axis is negative. When calculating area we do not use this convention, so the answer will have to be adjusted appropriately.

**Example 5.6.1:** Find the area contained between the quadratic
\[ y = 3 + 2x - x^2 \]
and the x-axis.

We have \( y = (3 - x)(x + 1) \), and from the graph we see that
\[
\text{area} = \int_{-1}^{3} 3 + 2x - x^2 \, dx \\
= \left[ 3x + x^2 - \frac{x^3}{3} \right]_{-1}^{3} - \frac{32}{3}
\]

Note that if the example had asked for the second and third arcs, we would have calculated
\[
\int_{2\pi}^{3\pi} \sin x \, dx - \int_{\pi}^{2\pi} \sin x \, dx.
\]

**Example 5.6.3:** Find the area enclosed by the line \( y = 2x \) and the curve
\[ y = 2x^2 - 3x^2. \]

Line and curve intersect when
\[ 2x^3 - 3x^2 = 2x \]
i.e. when
\[ x(2x + 1)(x - 2) = 0. \]

\[ x = 0, \quad x = -\frac{1}{2}, \quad x = 2. \]
Let \( y_1 = 2x \) and \( y_2 = 2x^3 - 3x^2 \). Then

\[
\text{area } A = \int_{-1}^{0} y_2 - y_1 \, dx = \int_{-1}^{0} 2x^3 - 3x^2 - 2x \, dx
\]

\[
= \left[ \frac{x^4}{2} - x^3 - x^2 \right]_{-1}^{0} = \frac{3}{32}
\]

and

\[
\text{area } B = \int_{0}^{1} y_2 - y_1 \, dx = \int_{0}^{1} 2x^3 + 3x^2 + 2x \, dx
\]

\[
= \left[ \frac{x^4}{2} + x^3 + x^2 \right]_{0}^{1} = 4.
\]

Therefore the total area is \( A + B = \frac{121}{32} \).

### 6. Real functions II

#### 6.1 Inverse trigonometric functions

We would like to define the inverse of \( \sin \), \( \cos \), and \( \tan \), to be denoted \( \sin^{-1} \), \( \cos^{-1} \), and \( \tan^{-1} \).

**Note:** (i) For these to be functions we have to restrict the range.

(ii) \( \sin^{-1} \) does not mean \( (\sin y)^{-1} \). This is an unfortunate problem with using \( \sin^{-1} = (\sin y)^{0} \). If \( n = -1 \) we must not do this!

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain</th>
<th>Range</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \sin^{-1} ) ( x )</td>
<td>( x \leq 1 )</td>
<td>( -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} )</td>
<td>( x = \sin y )</td>
</tr>
<tr>
<td>( y = \cos^{-1} ) ( x )</td>
<td>( x \leq 1 )</td>
<td>( 0 \leq y \leq \pi )</td>
<td>( x = \cos y )</td>
</tr>
<tr>
<td>( y = \tan^{-1} ) ( x )</td>
<td>( \mathbb{R} )</td>
<td>( -\frac{\pi}{2} &lt; y &lt; \frac{\pi}{2} )</td>
<td>( x = \tan y )</td>
</tr>
</tbody>
</table>

Note that \( \sin^{-1} \) and \( \tan^{-1} \) are increasing, odd functions, while \( \cos^{-1} \) is decreasing.

Sometimes we write \( \arcsin x \) for \( \sin^{-1} x \) and similarly \( \arccos x \) for \( \cos^{-1} x \) and \( \arctan x \) for \( \tan^{-1} x \).

**Example 6.1.1:** \( \alpha = \sin^{-1} \left( \frac{1}{2} \right) \) implies that \( \sin \alpha = \frac{1}{2} \) and \( -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2} \). Hence \( \alpha = \frac{\pi}{6} \).

**Proposition 6.1.3:** We have for \( -1 \leq x \leq 1 \) that

\[ \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x. \]

**Proof:** Let \( y = \sin^{-1} x \). Then \( x = \sin y \) with \( -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \). Therefore

\[
\cos^{-1} x = \frac{\pi}{2} - y = \frac{\pi}{2} - \sin^{-1} x.
\]

**Example 6.1.2:** Express \( \sin(2 \cos^{-1} x) \) in terms of \( x \) only.

Let \( y = \cos^{-1} x \). Then

\[
\sin(2 \cos^{-1} x) = \sin 2y = 2 \sin y \cos y.
\]

Now \( \cos^{-1} x = y \) gives \( \cos y = x \) with \( 0 \leq y \leq \pi \), and

\[
\sin^2 y = 1 - \cos^2 y = 1 - x^2.
\]

Note that \( \sin y \geq 0 \) as \( 0 \leq y \leq \pi \), and so

\[
\sin y = \sqrt{1 - x^2}.
\]

Therefore

\[
\sin(2 \cos^{-1} x) = 2x \sqrt{1 - x^2}.
\]

**Proposition 6.1.4:** We have

\[
\tan^{-1} a + \tan^{-1} b = \tan^{-1} \left( \frac{a + b}{1 - ab} \right) + pr
\]

where

\[
\rho = \begin{cases} 
-1 & \text{if } -\pi < \tan^{-1} a + \tan^{-1} b < -\frac{\pi}{2} \\
0 & \text{if } -\frac{\pi}{2} < \tan^{-1} a + \tan^{-1} b < \frac{\pi}{2} \\
1 & \text{if } \frac{\pi}{2} < \tan^{-1} a + \tan^{-1} b < \pi.
\end{cases}
\]

**Proof:** Let \( \alpha = \tan^{-1} a \) and \( \beta = \tan^{-1} b \), so \( -\frac{\pi}{2} < \alpha, \beta < \frac{\pi}{2} \) and \( \tan \alpha = a \) and \( \tan \beta = b \). We have

\[
\frac{a + b}{1 - ab} = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \tan(\alpha + \beta) = \tan(\alpha + \beta + m\pi)
\]

(for all \( n \in \mathbb{Z} \)) and \( -\pi < \alpha + \beta < \pi \). Now \( \tan^{-1} \left( \frac{a + b}{1 - ab} \right) \) must lie between \( -\frac{\pi}{2} \) and \( \frac{\pi}{2} \), and equal \( \alpha + \beta + m\pi \), for some value of \( n \). The result now follows by inspection.

□
Example 6.1.5: Find $u$ such that
\[
\tan^{-1} \frac{3}{4} + \tan^{-1} \frac{5}{12} = \tan^{-1} u.
\]
Let $\alpha = \tan^{-1} \frac{3}{4}$, so $\tan \alpha = \frac{3}{4}$ with $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$. Let $\beta = \tan^{-1} \frac{5}{12}$, so $\tan \beta = \frac{5}{12}$ with $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$. Clearly $0 < \alpha, \beta < \frac{\pi}{2}$ and so $0 < \alpha + \beta < \pi$. Hence by the last Proposition we have
\[
\tan^{-1} \frac{3}{4} + \tan^{-1} \frac{5}{12} = \tan^{-1} \left( \frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4} \cdot \frac{5}{12}} \right) = \tan^{-1} \frac{56}{33}.
\]

Now suppose that $x > 1$, i.e. $\frac{x}{2} < \tan^{-1} x < \frac{\pi}{2}$. Then $-1 < \frac{1}{1+x^2} < 0$, so $-\frac{\pi}{2} < \tan^{-1} \frac{1}{1+x^2} < 0$. Hence for $x > 1$ we have
\[
0 < \tan^{-1} x + \tan^{-1} \frac{1-x}{1+x} = \tan^{-1} \frac{\frac{x}{2} + \frac{1}{1+x^2}}{1 - \frac{x}{2} \cdot \frac{1}{1+x^2}}.
\]
Thus for all $x \geq 0$ we have
\[
\tan^{-1} x + \tan^{-1} \frac{1-x}{1+x} = \tan^{-1} \frac{x + \frac{1}{1+x^2}}{1 - x \cdot \frac{1}{1+x^2}} = \tan^{-1}(1) + \frac{\pi}{4}.
\]

Let $y = \cos^{-1} x$. Then $\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$ and hence
\[
\frac{dy}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}.
\]
Finally let $y = \tan^{-1} x$. By definition $x = \tan y$ with $-\frac{\pi}{2} < y < \frac{\pi}{2}$. We differentiate with respect to $x$:
\[
\sec^2 \frac{dy}{dx} = 1 \quad \Rightarrow 
\frac{dy}{dx} = \frac{1}{\sec^2 y}.
\]
Now $\sec^2 y = 1 + \tan^2 y$ and so $\sec^2 y = 1 + x^2$. Thus we have shown that
\[
\frac{dy}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}.
\]

6.3 Integration and inverse trigonometric functions
First suppose that $y = \sin^{-1} (x/a)$. Then
\[
\frac{dy}{dx} = \frac{1}{\sqrt{1-(x/a)^2}} \cdot \frac{1}{a} = \frac{1}{\sqrt{a^2 - x^2}}.
\]
Hence
\[
\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left( \frac{x}{a} \right) + C.
\]

Next suppose that $y = \tan^{-1}(x/a)$. Then
\[
\frac{dy}{dx} = \frac{1}{1+(x/a)^2} \cdot \frac{1}{a} = \frac{1}{\sqrt{a^2 + x^2}}.
\]
Hence
\[
\int \frac{1}{\sqrt{a^2 + x^2}} dx = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C.
\]
We can now integrate rational functions with quadratic denominators.

Example 6.1.6: Simplify
\[
\tan^{-1} x + \tan^{-1} \frac{1-x}{1+x}
\]
for $x \geq 0$.
First suppose that $0 \leq x \leq 1$, i.e. $0 \leq \tan^{-1} x \leq \frac{\pi}{4}$. Then
\[
\frac{1}{1+x} = \frac{1}{1-\frac{1}{x}} \Rightarrow \text{so } 0 \leq \frac{1}{1+x} \leq 1; \text{i.e.}
\]
\[
0 \leq \tan^{-1} \frac{1-x}{1+x} \leq \frac{\pi}{4}.
\]
Hence for $0 \leq x \leq 1$ we have
\[
0 \leq \tan^{-1} x + \tan^{-1} \frac{1-x}{1+x} \leq \frac{\pi}{2}.
\]

6.2 Differentiation of inverse trigonometric functions
Let $y = \sin^{-1} x$. By definition $x = \sin y$ with $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. We differentiate with respect to $x$:
\[
\cos \frac{dy}{dx} = 1 \quad \Rightarrow 
\frac{dy}{dx} = \frac{1}{\cos y}.
\]
Now $\cos^2 y = 1 - \sin^2 y$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, hence $\cos y = +\sqrt{1 - \sin^2 y}$. Thus we have shown that
\[
\frac{dx}{dy}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}
\]

Example 6.2.1: Differentiate $\sin^{-1}(\sqrt{x})$.
Let $y = \sin^{-1} u$ with $u = x^{1/2}$, so $\frac{dy}{du} = \frac{1}{2} x^{-1/2}$. Then
\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2 \sqrt{x}} \cdot \frac{1}{\sqrt{1-x\cdot x}} = \frac{1}{2 \sqrt{1-x}}.
\]

Example 6.2.2: Differentiate $\tan^{-1}(2x + 1)$.
Let $y = \tan^{-1} (2x + 1)$. Then
\[
\frac{dy}{dx} = \frac{2}{1 + (2x+1)^2} = \frac{2}{4x^2 + 4x + 2} = \frac{1}{2x^2 + 2x + 1}.
\]
Example 6.3.1: Integrate
\[ \int \frac{1}{x^2 + 2x + 5} \, dx. \]
The denominator does not factorise, so we complete the square.
\[ \int \frac{1}{x^2 + 2x + 5} \, dx = \int \frac{1}{(x + 1)^2 + 4} \, dx = \frac{1}{2} \tan^{-1} \left( \frac{x + 1}{2} \right) + C. \]

Example 6.3.2: Integrate
\[ \int \frac{x + 3}{x^2 + 2x + 5} \, dx. \]
Note that \( \frac{d}{dx}(x^2 + 2x + 5) = 2x + 2. \) Thus
\[
\int \frac{x + 3}{x^2 + 2x + 5} \, dx = \frac{1}{2} \int \frac{2x + 2}{x^2 + 2x + 5} \, dx + 2 \int \frac{1}{(x + 1)^2 + 4} \, dx \\
= \frac{1}{2} \ln(x^2 + 2x + 5) + \tan^{-1} \left( \frac{x + 1}{2} \right) + C.
\]

Example 6.3.3:
\[
\int \frac{1}{2x^2 + 2x + 1} \, dx = \int \frac{1}{2(x^2 + x + \frac{1}{2})} \, dx \\
= \frac{1}{2} \int \frac{1}{(x + \frac{1}{2})^2 + \frac{1}{4}} \, dx \\
= \frac{1}{2} \left( \frac{1}{\sqrt{2}} \right) \tan^{-1} \left( x + \frac{1}{2} \right) + C \\
= \tan^{-1}(2x + 1) + C.
\]

We can also deal with more complicated rational functions by using these methods together with partial fractions.

Finally, we consider the integrals of inverse trigonometric functions. To integrate \( \sin^{-1} x \) we use integration by parts with \( u = \sin^{-1} x \) and \( v = x \).
\[
\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1 - x^2}} \, dx = x \sin^{-1} x + \sqrt{1 - x^2} + C.
\]
Similarly
\[
\int \tan^{-1} x \, dx = x \tan^{-1} x - \int \frac{x}{x^2 + 1} \, dx = x \tan^{-1} x - \frac{1}{2} \ln(x^2 + 1) + C.
\]