We saw in Example 5.4.2 that we sometimes need to apply integration by parts several times in the course of a single calculation.

Example 5.4.4: For $n \geq 0$ let

$$S_n = \int x^n \cos 2x \, dx.$$

Find an expression for S_n in terms of S_{n-2}, and hence evaluate S_4.

Let $u = x^n$ and $\frac{dv}{dx} = \cos 2x$. Then $\frac{du}{dx} = nx^{n-1}$ and $v = \frac{1}{2}\sin(2x)$.

Integrating by parts we have

$$\begin{aligned} \int x^n \cos 2x \, dx &= \frac{x^n}{2}\sin(2x) - \int \frac{n}{2}x^{n-1}\sin 2x \, dx \\ &= \frac{x^n}{2}\sin(2x) + \frac{n}{4}x^{n-1}\cos 2x \\ &\quad - \int \frac{n(n-1)}{4}x^{n-2}\cos 2x \, dx \\ &= \frac{x^n}{2}\sin(2x) + \frac{n}{4}x^{n-1}\cos 2x - \frac{n(n-1)}{4}S_{n-2} \end{aligned}$$

where the second equality follows using integration by parts with $u = \frac{n}{2}x^{n-1}$ and $\frac{dv}{dx} = \sin 2x$. Thus we have found a formula for S_n in terms of S_{n-2}.

Clearly $S_0 = \int \cos 2x \, dx = \frac{1}{2}\sin 2x + C$. Hence

$$S_2 = \frac{x^2}{2}\sin(2x) + \frac{2}{4}x\cos 2x - \frac{1}{4}\sin 2x + C'$$

for some constant C' and

$$\begin{aligned} S_4 &= \frac{x^4}{2}\sin(2x) + \frac{4}{4}x^3\cos 2x \\ &\quad -3\left(\frac{x^2}{2}\sin(2x) + \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x + C'\right) \\ &= \frac{1}{4}(2x^4 - 6x^2 + 3)\sin 2x + \frac{1}{2}(2x^3 - 3x)\cos 2x + C'' \end{aligned}$$

for some constant C''.

In some examples integration by parts does not lead to a simpler integral. However, even in these cases we can sometimes use this method to solve the original problem.

Example 5.4.5: Calculate

$$\int e^x \cos x \, dx.$$

Let $u = e^x$ and $\frac{dv}{dx} = \cos x$. Integrating by parts we obtain

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

Integrating by parts again we have

$$\int e^x \cos x \, dx = e^x \sin x - \left[-e^x \cos x + \int e^x \cos x \, dx\right].$$

The final integral is identical to that we first wished to calculate, however we can now rearrange this formula to obtain

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x$$

from which we deduce that

$$\int e^x \cos x \, dx = \frac{1}{2}(e^x \sin x + e^x \cos x)$$

5.5 The definite integral

If

$$\int g(x)\, dx = G(x) + C$$

then we define

$$\int_a^b g(x)\, dx = G(b) - G(a)$$

which we also denote by

$$\Big[G(x)\Big]_a^b.$$

Example 5.5.1:

$$\begin{aligned} \int_1^4 \frac{1}{(x+3)^2} dx &= \left[\frac{-1}{x+3}\right]_1^4 \\ &= -\frac{1}{7} - \left(-\frac{1}{4}\right) = \frac{3}{28.} \end{aligned}$$

In the next example we will apply Example 5.2.3.

Example 5.5.2:

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin^4 x \cos x \, dx &= \left[\frac{1}{5}\sin^5 x\right]_0^{\frac{\pi}{2}} \\ &= \frac{1}{5} - 0 = \frac{1}{5}. \end{aligned}$$

When integrating a definite integral by substitution we must be careful to convert the limits into the new variable.

Example 5.5.3: Calculate

$$\int_0^2 \sqrt{4-x^2}\,dx.$$

Let $x = 2\sin\theta$, so $\frac{dx}{d\theta} = 2\cos\theta$. We have

$$4 - x^2 = 4 - 4\sin^2\theta = 4\cos^2\theta$$

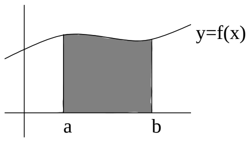
and in changing variable we have

$$\begin{aligned} x = 0 &\longrightarrow \theta = 0 \\ x = 2 &\longrightarrow \theta = \frac{\pi}{2}. \end{aligned}$$

Combining these observations we obtain

$$\begin{aligned} \int_0^2 \sqrt{4-x^2}\,dx &= \int_0^{\frac{\pi}{2}} 2\cos\theta\, 2\cos\theta\, d\theta \\ &= \int_0^{\frac{\pi}{2}} 4\cos^2\theta\, d\theta \\ &= \int_0^{\frac{\pi}{2}} 2(1+\cos 2\theta)\, d\theta \\ &= \left[2\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{\frac{\pi}{2}} \\ &= 2\left[\frac{\pi}{2} + 0 - 0 - 0\right] = \pi. \end{aligned}$$

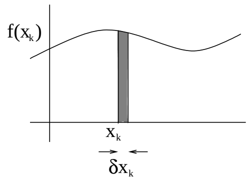
5.6 Integration as a measure of content



The area contained between the curve $y = f(x)$, the lines $x = a$ and $x = b$ (for $a < b$) and the x-axis is given by

$$\int_a^b f(x)\,dx.$$

This follows from the definition of integration as a measure:

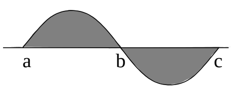


$$\text{area} \approx \sum_{k=1}^{n} f(x_k)\delta x_k$$

and the fundamental theorem of calculus which states that this definition agrees with that coming from the antiderivative.

Note that this result relies on the convention that area below the x-axis is negative. When calculating area we do not use this convention, so the answer will have to be adjusted appropriately.

So for



we have

$$\int_a^b f(x)\,dx = -\int_b^c f(x)\,dx$$

although the total area is clearly non-zero.

If $b < a$ we define

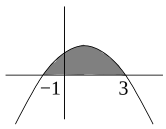
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

(This must clearly be the case from the definition of integration using the antiderivative.)

Example 5.6.1: Find the area contained between the quadratic

$$y = 3 + 2x - x^2$$

and the x-axis.



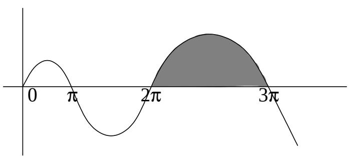
We have $y = (3-x)(x+1)$, and from the graph we see that

$$\begin{aligned} \text{area} &= \int_{-1}^{3} 3 + 2x - x^2\,dx \\ &= \left[3x + x^2 - \frac{x^3}{3}\right]_{-1}^{3} = \frac{32}{3}. \end{aligned}$$

Example 5.6.2: Find the area contained in the third arc of the curve

$$y = x\sin x$$

for $x \geq 0$.



$$\begin{aligned} \text{area} &= \int_{2\pi}^{3\pi} x\sin x\,dx \qquad \text{(use integration by parts)} \\ &= \left[-x\cos x - \int(-\cos x)\,dx\right]_{2\pi}^{3\pi} = \left[-x\cos x + \sin x\right]_{2\pi}^{3\pi} \\ &= [(-3\pi)(-1) + 0] - [(-2\pi)(1) + 0] = 5\pi. \end{aligned}$$

Note that if the example had asked for the second and third arcs, we would have calculated

$$\int_{2\pi}^{3\pi} x\sin x\,dx - \int_{\pi}^{2\pi} x\sin x\,dx.$$

Example 5.6.3: Find the area enclosed by the line $y = 2x$ and the curve

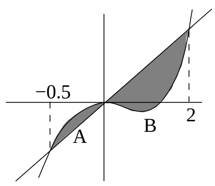
$$y = 2x^3 - 3x^2.$$

Line and curve intersect when

$$2x^3 - 3x^2 = 2x$$

i.e. when

$$x(2x+1)(x-2) = 0.$$



Let $y_1 = 2x$ and $y_2 = 2x^3 - 3x^2$. Then

$$\begin{aligned}\text{area A} &= \int_{-\frac{1}{2}}^{0} y_2 - y_1\,dx = \int_{-\frac{1}{2}}^{0} 2x^3 - 3x^2 - 2x\,dx \\ &= \left[\frac{x^4}{2} - x^3 - x^2\right]_{-\frac{1}{2}}^{0} = \frac{3}{32}.\end{aligned}$$

and

$$\begin{aligned}\text{area B} &= \int_0^2 y_1 - y_2\,dx = \int_0^2 -2x^3 + 3x^2 + 2x\,dx \\ &= \left[-\frac{x^4}{2} + x^3 + x^2\right]_0^2 = 4.\end{aligned}$$

Therefore the total area is $A + B = \frac{131}{32}$.

6. Real functions II

6.1 Inverse trigonometric functions

We would like to define the inverse of sin, cos, and tan, to be denoted $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$.

Note: (i) For these to be functions we have to restrict the range.
(ii) $\sin^{-1} y$ does not mean $(\sin y)^{-1}$. This is an unfortunate problem with using $\sin^n y = (\sin y)^n$. If $n = -1$ we must not do this!

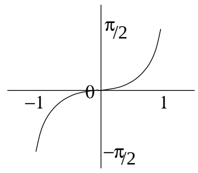
Function	Domain	Range	Definition
$y = \sin^{-1} x$	$\lvert x\rvert \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$	$x = \sin y$
$y = \cos^{-1} x$	$\lvert x\rvert \le 1$	$0 \le y \le \pi$	$x = \cos y$
$y = \tan^{-1} x$	$\mathbb{R}$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$	$x = \tan y$

Note that $\sin^{-1}$ and $\tan^{-1}$ are increasing, odd functions, while $\cos^{-1}$ is decreasing.

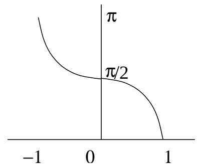
Sometimes we write arcsin x for $\sin^{-1} x$ and similarly arccos x for $\cos^{-1} x$ and arctan x for $\tan^{-1} x$.

The graphs of these functions are:

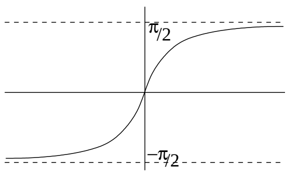
$y = \sin^{-1} \theta$



$y = \cos^{-1} \theta$



$y = \tan^{-1} \theta$



Example 6.1.1: $\alpha = \sin^{-1}(\frac{1}{2})$ implies that $\sin\alpha = \frac{1}{2}$ and $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$. Hence $\alpha = \frac{\pi}{6}$.

Example 6.1.2: Express $\sin(2\cos^{-1} x)$ in terms of x only.

Let $y = \cos^{-1} x$. Then

$$\sin(2\cos^{-1} x) = \sin 2y = 2\sin y \cos y.$$

Now $\cos^{-1} x = y$ gives $\cos y = x$ with $0 \le y \le \pi$, and

$$\sin^2 y = 1 - \cos^2 y = 1 - x^2.$$

Note that $\sin y \ge 0$ as $0 \le y \le \pi$, and so

$$\sin y = \sqrt{1 - x^2}.$$

Therefore

$$\sin(2\cos^{-1} x) = 2x\sqrt{1 - x^2}.$$

Proposition 6.1.3: *We have for $-1 \le x \le 1$ that*

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x.$$

Proof: Let $y = \sin^{-1} x$.
Then $x = \sin y$ with $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, and $x = \cos(\frac{\pi}{2} - y)$ where $0 \le \frac{\pi}{2} - y \le \pi$. Therefore

$$\cos^{-1} x = \frac{\pi}{2} - y = \frac{\pi}{2} - \sin^{-1} x.$$

$\square$

Proposition 6.1.4: *We have*

$$\tan^{-1} a + \tan^{-1} b = \tan^{-1}\left(\frac{a+b}{1-ab}\right) + p\pi$$

where

$$p = \begin{cases} -1 & \text{if } -\pi < \tan^{-1} a + \tan^{-1} b < -\frac{\pi}{2} \\ 0 & \text{if } -\frac{\pi}{2} < \tan^{-1} a + \tan^{-1} b < \frac{\pi}{2} \\ 1 & \text{if } \frac{\pi}{2} < \tan^{-1} a + \tan^{-1} b < \pi. \end{cases}$$

Proof: Let $\alpha = \tan^{-1} a$ and $\beta = \tan^{-1} b$, so $-\frac{\pi}{2} < \alpha, \beta < \frac{\pi}{2}$ and $\tan\alpha = a$ and $\tan\beta = b$. We have

$$\frac{a+b}{1-ab} = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \tan(\alpha+\beta) = \tan(\alpha+\beta+n\pi)$$

(for all $n \in \mathbb{Z}$) and $-\pi < \alpha + \beta < \pi$. Now $\tan^{-1}\left(\frac{a+b}{1-ab}\right)$ must lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and equal $\alpha + \beta + n\pi$, for some value of n. The result now follows by inspection. $\square$

Example 6.1.5: Find u such that

$$\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{5}{12}=\tan^{-1}u.$$

Let $\alpha=\tan^{-1}\frac{3}{4}$, so $\tan\alpha=\frac{3}{4}$ with $-\frac{\pi}{2}<\alpha<\frac{\pi}{2}$. Let $\beta=\tan^{-1}\frac{5}{12}$, so $\tan\beta=\frac{5}{12}$ with $-\frac{\pi}{2}<\beta<\frac{\pi}{2}$. Clearly $0<\alpha,\beta<\frac{\pi}{4}$ and so $0<\alpha+\beta<\frac{\pi}{2}$. Hence by the last Proposition we have

$$\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{5}{12}=\tan^{-1}\left(\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\cdot\frac{5}{12}}\right)=\tan^{-1}\frac{56}{33}.$$

Example 6.1.6: Simplify

$$\tan^{-1}x+\tan^{-1}\frac{1-x}{1+x}$$

for $x\geq 0$.

First suppose that $0\leq x\leq 1$, i.e $0\leq\tan^{-1}x\leq\frac{\pi}{4}$. Then $\frac{1-x}{1+x}=-1+\frac{2}{1+x}$ and so $0\leq\frac{1-x}{1+x}\leq 1$; i.e.

$$0\leq\tan^{-1}\frac{1-x}{1+x}\leq\frac{\pi}{4}.$$

Hence for $0\leq x\leq 1$ we have

$$0\leq\tan^{-1}x+\tan^{-1}\frac{1-x}{1+x}\leq\frac{\pi}{2}.$$

Now suppose that $x>1$, i.e $\frac{\pi}{4}<\tan^{-1}x<\frac{\pi}{2}$. Then $-1<\frac{1-x}{1+x}<0$, so $-\frac{\pi}{4}<\tan^{-1}\frac{1-x}{1+x}<0$. Hence for $x>1$ we have

$$0<\tan^{-1}x+\tan^{-1}\frac{1-x}{1+x}<\frac{\pi}{2}.$$

Thus for all $x\geq 0$ we have

$$\tan^{-1}x+\tan^{-1}\frac{1-x}{1+x}=\tan^{-1}\left(\frac{x+\frac{1-x}{1+x}}{1-x\left(\frac{1-x}{1+x}\right)}\right)=\tan^{-1}\left(\frac{x^2+1}{1+x^2}\right)$$
$$=\tan^{-1}(1)=\frac{\pi}{4}.$$

6.2 Differentiation of inverse trigonometric functions

Let $y=\sin^{-1}x$. By definition $x=\sin y$ with $-\frac{\pi}{2}\leq y\leq\frac{\pi}{2}$. We differentiate with respect to x:

$$\cos y\frac{dy}{dx}=1\qquad\text{so}\qquad\frac{dy}{dx}=\frac{1}{\cos y}.$$

Now $\cos^2 y=1-\sin^2 y$ and $-\frac{\pi}{2}\leq y\leq\frac{\pi}{2}$, hence $\cos y=+\sqrt{1-\sin^2 y}$. Thus we have shown that

$$\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}.$$

Let $y=\cos^{-1}x$. Then $\cos^{-1}x=\frac{\pi}{2}-\sin^{-1}x$ and hence

$$\frac{d}{dx}(\cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}}.$$

Finally let $y=\tan^{-1}x$. By definition $x=\tan y$ with $-\frac{\pi}{2}<y<\frac{\pi}{2}$. We differentiate with respect to x:

$$\sec^2 y\frac{dy}{dx}=1\qquad\text{so}\qquad\frac{dy}{dx}=\frac{1}{\sec^2 y}.$$

Now $\sec^2 y=1+\tan^2 y$ and so $\sec^2 y=1+x^2$. Thus we have shown that

$$\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}.$$

Example 6.2.1: Differentiate $\sin^{-1}(\sqrt{x})$.

Let $y=\sin^{-1}u$ with $u=x^{\frac{1}{2}}$, so $\frac{du}{dx}=\frac{1}{2}x^{-\frac{1}{2}}$. Then

$$\frac{dy}{du}=\frac{1}{\sqrt{1-u^2}}$$

and

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=\frac{1}{\sqrt{1-x}}\frac{1}{2\sqrt{x}}=\frac{1}{2\sqrt{x(1-x)}}.$$

Example 6.2.2: Differentiate $\tan^{-1}(2x+1)$.

Let $y=\tan^{-1}(2x+1)$. Then

$$\frac{dy}{dx}=\frac{2}{1+(2x+1)^2}=\frac{2}{4x^2+4x+2}=\frac{1}{2x^2+2x+1}.$$

6.3 Integration and inverse trigonometric functions

First suppose that $y=\sin^{-1}(x/a)$. Then

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-(\frac{x}{a})^2}}.\frac{1}{a}=\frac{1}{\sqrt{a^2-x^2}}.$$

Hence

$$\int\frac{1}{\sqrt{a^2-x^2}}\,dx=\sin^{-1}\left(\frac{x}{a}\right)+C.$$

Next suppose that $y=\tan^{-1}(x/a)$. Then

$$\frac{dy}{dx}=\frac{1}{1+(\frac{x}{a})^2}.\frac{1}{a}=\frac{a}{a^2+x^2}.$$

Hence

$$\int\frac{1}{x^2+a^2}\,dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C.$$

We can now integrate rational functions with quadratic denominators.

Example 6.3.1: Integrate

$$\int \frac{1}{x^2+2x+5}\,dx.$$

The denominator does not factorise, so we complete the square.

$$\int \frac{1}{x^2+2x+5}\,dx = \int \frac{1}{(x+1)^2+4}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C.$$

Example 6.3.2: Integrate

$$\int \frac{x+3}{x^2+2x+5}\,dx.$$

Note that $\frac{\mathrm{d}}{\mathrm{d}x}(x^2+2x+5)=2x+2$. Thus

$$\begin{aligned}\int \frac{x+3}{x^2+2x+5}\,dx &= \int \frac{\frac{1}{2}(2x+2)+2}{x^2+2x+5}\,dx\\ &= \frac{1}{2}\int \frac{2x+2}{x^2+2x+5}\,dx + 2\int \frac{1}{(x+1)^2+4}\,dx\\ &= \frac{1}{2}\ln(x^2+2x+5)+\tan^{-1}\left(\frac{x+1}{2}\right)+C.\end{aligned}$$

Example 6.3.3:

$$\begin{aligned}\int \frac{1}{2x^2+2x+1}\,dx &= \int \frac{1}{2(x^2+x+\frac{1}{2})}\,dx\\ &= \frac{1}{2}\int \frac{1}{(x+\frac{1}{2})^2+\frac{1}{4}}\,dx\\ &= \frac{1}{2}\left(\frac{1}{\frac{1}{2}}\right)\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{1}{2}}\right)+C\\ &= \tan^{-1}(2x+1)+C.\end{aligned}$$

(Compare with Ex 6.2.2.)

We can also deal with more complicated rational functions by using these methods together with partial fractions.

Finally, we consider the integrals of inverse trigonometric functions. To integrate $\sin^{-1}x$ we use integration by parts with $u=\sin^{-1}x$ and $v=x$.

$$\int \sin^{-1}x = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx = x\sin^{-1}x+\sqrt{1-x^2}+C.$$

Similarly

$$\int \tan^{-1}x = x\tan^{-1}x - \int \frac{x}{x^2+1}\,dx = x\tan^{-1}x - \frac{1}{2}\ln(x^2+1)+C.$$