6.4 Integration using $\tan(x/2)$

We will revisit the double angle identities:

$$
\begin{aligned}
\sin x &= 2\sin(x/2)\cos(x/2) \\
&= \frac{2\tan(x/2)}{\sec^2(x/2)} = \frac{2\tan(x/2)}{1+\tan^2(x/2)} \\
\cos x &= \cos^2(x/2) - \sin^2(x/2) \\
&= \frac{1-\tan^2(x/2)}{\sec^2(x/2)} = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)} \\
\tan x &= \frac{2\tan(x/2)}{1-\tan^2(x/2)}.
\end{aligned}
$$

So writing $t = \tan(x/2)$ we have

$$\sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad \tan x = \frac{2t}{1-t^2}.$$

Also

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}\sec^2(x/2) = \frac{1}{2}(1+\tan^2(x/2)) = \frac{1+t^2}{2}$$

so

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2}{1+t^2}.$$

We can use these formulas to calculate integrals of the form

$$\int \frac{1}{a\cos x + b\sin x + c}\,dx$$

by converting them into integrals of rational functions.

Example 6.4.1: Integrate

$$\int \frac{1}{12+13\sin x}\,dx.$$

Let $t = \tan(x/2)$. Then

$$
\begin{aligned}
\int \frac{1}{12+13\sin x}\,dx &= \int \frac{1}{\left(12+13\frac{2t}{1+t^2}\right)}\frac{2}{1+t^2}\,dt \\
&= \int \frac{1}{6t^2+13t+6}\,dt = \int \frac{1}{(3t+2)(2t+3)}\,dt \\
&= \frac{1}{5}\int \frac{3}{3t+2} - \frac{2}{2t+3}\,dt \\
&= \frac{1}{5}(\ln(3t+2) - \ln(2t+3)) + C \\
&= \frac{1}{5}(\ln(3\tan(x/2)+2) - \ln(2\tan(x/2)+3)) + C
\end{aligned}
$$

6.5 Hyperbolic functions

We define the hyperbolic cosine of x by

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

and the hyperbolic sine of x by

$$\sinh x = \frac{1}{2}(e^x - e^{-x}).$$

These functions turn out to be very similar (in certain respects) to the usual trigonometric functions. For example, they satisfy similar identities. This will be justified more precisely when we consider complex numbers next term.

By analogy with the standard trig functions we define

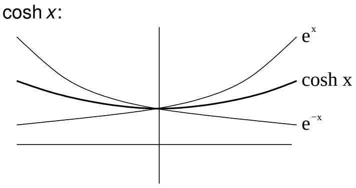
$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \operatorname{sech} x = \frac{1}{\cosh x} \qquad \operatorname{cosech} x = \frac{1}{\sinh x}$$

and

$$\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}.$$

Although these functions are in some ways very similar to the standard trig functions, they also have some striking differences. For example, they are not periodic.

The graph of $\cosh x$:

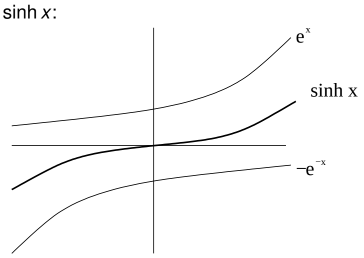


This is an even function, and $\cosh 0 = 1$. Note that this is also the minimum value of cosh: if $y = \cosh x$ then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x - e^{-x}}{2}$$

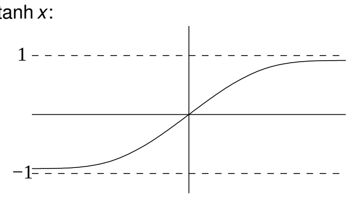
so $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ implies that $e^x - e^{-x} = 0$, i.e. $e^{2x} = 1$, so $x = 0$.

The graph of $\sinh x$:



This is an odd function, and $\sinh 0 = 0$. There are no stationary points, but there is a point of inflection at 0.

The graph of $\tanh x$:



Note that the domain of all three functions is $\mathbb{R}$. The range of sinh is $\mathbb{R}$, of cosh is $y \geq 1$, and of tanh is $|y| < 1$.

Last time we claimed that hyperbolic functions had many similarities with trigonometric functions — but saw that their graphs were quite different. To justify, in part, our claim, we will now consider various hyperbolic identities.

Example 6.5.1: Show that

$$\sinh 2x = 2\sinh x \cosh x.$$

$$\begin{aligned} 2\sinh x \cosh x &= 2\frac{1}{2}(e^x - e^{-x})\frac{1}{2}(e^x + e^{-x}) \\ &= \frac{1}{2}(e^{2x} - 1 + 1 - e^{-2x}) = \sinh(2x). \end{aligned}$$

Example 6.5.2: Show that

$$\cosh^2 x - \sinh^2 x = 1.$$

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= \frac{1}{4}(e^{2x} + 2 + e^{-2x}) - \frac{1}{4}(e^{2x} - 2 + e^{-2x}) \\ &= \frac{4}{4} = 1. \end{aligned}$$

The last two examples are both very similar to the corresponding trig formulas, apart from the minus sign in 6.5.2. This is generally true: we can find new hyperbolic identities using

Osborn's Rule: (i) Change each trig function in an identity to the corresponding hyperbolic function.
(ii) Whenever a product of two sines occurs, change the sign of that term.

This rule does not prove the identity; it can only be used to suggest possible identities, which can then be verified. Also note that products of sines can be disguised: for example in $\tan^2 x$ we have $\frac{\sin^2 x}{\cos^2 x}$.

Example 6.5.3: Find a hyperbolic analogue to

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}.$$

Osborn's rule suggests that we try

$$\tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x}.$$

The righthand side equals

$$\frac{2\sinh x}{\cosh x}\frac{1}{1 + \frac{\sinh^2 x}{\cosh^2 x}} = \frac{2\sinh x}{\cosh x}\frac{\cosh^2 x}{\cosh^2 x + \sinh^2 x} = \frac{2\sinh x \cosh x}{\cosh^2 x + \sinh^2 x}.$$

By Example 6.5.1 this equals

$$\frac{\sinh 2x}{\cosh^2 x + \sinh^2 x}$$

so it is enough to prove that

$$\cosh^2 x + \sinh^2 x = \cosh 2x.$$

But

$$\begin{aligned} \cosh^2 x + \sinh^2 x &= \frac{1}{4}(e^{2x} + 2 + e^{-2x}) + \frac{1}{4}(e^{2x} - 2 + e^{-2x}) \\ &= \frac{1}{2}(e^{2x} + e^{-2x}) = \cosh 2x \end{aligned}$$

as required.

6.6 Solving hyperbolic equations

These are usually simpler to solve than the corresponding trig equations.

Example 6.6.1: Solve

$$3\sinh x - \cosh x = 1.$$

We have

$$\frac{3}{2}(e^x - e^{-x}) - \frac{1}{2}(e^x + e^{-x}) = 1$$

which becomes

$$e^x - 2e^{-x} = 1.$$

Therefore

$$e^{2x} - e^x - 2 = 0$$

or

$$(e^x + 1)(e^x - 2) = 0.$$

$e^x = -1$ is impossible, so the only solution is $e^x = 2$, i.e. $x = \ln 2$.

Sometimes, as for standard trig functions, it is best to use an identity to simplify the equation.

Example 6.6.1: Solve

$$12\cosh^2 x + 7\sinh x = 24.$$

We use $\cosh^2 x - \sinh^2 x = 1$. Then we have

$$12(1 + \sinh^2 x) + 7\sinh x = 24$$

which simplifies to

$$(3\sinh x + 4)(4\sinh x - 3) = 0.$$

So $\sinh x = -\frac{4}{3}$ or $\sinh x = \frac{3}{4}$.

If $\sinh x = -\frac{4}{3}$ then
$$\frac{e^x - e^{-x}}{2} = -\frac{4}{3}$$
i.e. $3e^x - 3e^{-x} = -8$, or equivalently $3e^{2x} + 8e^x - 3 = 0$. Therefore
$$(3e^x - 1)(e^x + 3) = 0$$
and hence $e^x = \frac{1}{3}$ (as $e^x = -3$ is impossible).
So $x = \ln \frac{1}{3} = -\ln 3$.

If $\sinh x = \frac{3}{4}$ then a similar calculation shows that $x = \ln 2$, and so the solutions to the equation are
$$x = -\ln 3 \qquad \text{and} \qquad x = \ln 2.$$

Integrate
$$\int \frac{x+3}{x^2+2x+5}\,dx.$$
Note that $\frac{d}{dx}(x^2+2x+5) = 2x+2$. Thus
$$\begin{aligned}\int \frac{x+3}{x^2+2x+5}\,dx &= \int \frac{\frac{1}{2}(2x+2)+2}{x^2+2x+5}\,dx \\ &= \frac{1}{2}\int \frac{2x+2}{x^2+2x+5}\,dx + 2\int \frac{1}{(x+1)^2+4}\,dx \\ &= \frac{1}{2}\ln(x^2+2x+5) + \tan^{-1}\left(\frac{x+1}{2}\right) + C.\end{aligned}$$

Example 6.3.3:
$$\begin{aligned}\int \frac{1}{2x^2+2x+1}\,dx &= \int \frac{1}{2(x^2+x+\frac{1}{2})}\,dx \\ &= \frac{1}{2}\int \frac{1}{(x+\frac{1}{2})^2+\frac{1}{4}}\,dx \\ &= \frac{1}{2}\left(\frac{1}{\frac{1}{2}}\right)\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{1}{2}}\right) + C \\ &= \tan^{-1}(2x+1) + C.\end{aligned}$$
(Compare with Ex 6.2.2.)

We can also deal with more complicated rational functions by using these methods together with partial fractions.

Finally, we consider the integrals of inverse trigonometric functions. To integrate $\sin^{-1} x$ we use integration by parts with $u = \sin^{-1} x$ and $v = x$.
$$\int \sin^{-1} x = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,dx = x\sin^{-1} x + \sqrt{1-x^2} + C.$$
Similarly
$$\int \tan^{-1} x = x\tan^{-1} x - \int \frac{x}{x^2+1}\,dx = x\tan^{-1} x - \frac{1}{2}\ln(x^2+1) + C.$$

6.7 Calculus of hyperbolics

It is easy to determine the derivatives of hyperbolic functions.

Example 6.7.1: Show that
$$\frac{d}{dx}(\cosh x) = \sinh x.$$
$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{1}{2}(e^x + e^{-x})\right) = \frac{1}{2}(e^x - e^{-x}) = \sinh x.$$
Similarly we can show that
$$\frac{d}{dx}(\sinh x) = \cosh x.$$
Note: Osborn's Rule does not apply to calculus.

We can now determine the derivatives of all the other hyperbolic functions. These should be memorised.

$f(x)$	$f'(x)$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\text{sech}^2\, x$
$\text{cosech}\, x$	$-\coth x \,\text{cosech}\, x$
$\coth x$	$-\text{cosech}^2\, x$
$\text{sech}\, x$	$-\text{sech}\, x \tanh x.$

Reversing the roles of the two columns (and remembering to add in the constant!) we can deduce the integrals of the functions in the right-hand column.

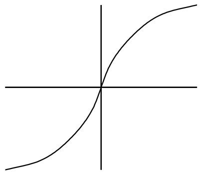
6.8 Inverse hyperbolic functions

First consider sinh. From the graph we see that this is injective with image $\mathbb{R}$. Thus it possesses an inverse function for all values of x. For $x \in \mathbb{R}$ we define
$$y = \sinh^{-1} x \quad \text{if and only if} \quad x = \sinh y.$$
Next consider tanh. This is also injective, but with image set $-1 < x < 1$. So for $-1 < x < 1$ we define
$$y = \tanh^{-1} x \quad \text{if and only if} \quad x = \tanh y.$$

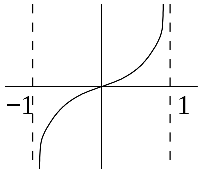
The function cosh is not injective, so we cannot define an inverse to the entire function. However, if we only consider $\cosh y$ on the domain $y \geq 0$ then the function is injective, with image set $\cosh y \geq 1$.

So for $x \geq 1$ we define
$$y = \cosh^{-1} x \quad \text{if and only if} \quad x = \cosh y \text{ and } y \geq 0.$$
We can sketch the graphs of these functions:

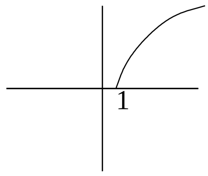
$y = \sinh^{-1} x$



$y = \tanh^{-1} x$



$y = \cosh^{-1} x$



Sometimes these functions are denoted by arsinh, arcosh, and artanh.
It is easy to differentiate these functions.

Example 6.8.1: Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}.$$

If $y = \sinh^{-1} x$ then $x = \sinh y$. Now

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \cosh y, \qquad \text{so} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cosh y}.$$

By Ex 6.5.2, and the fact that $\cosh y \geq 0$ for all y, we have that

$$\cosh y = \sqrt{\sinh^2 y + 1} = \sqrt{x^2+1}.$$

So

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}.$$

Similarly

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}.$$

Example 6.8.2: Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tanh^{-1} x) = \frac{1}{1-x^2}.$$

If $y = \tanh^{-1} x$ then $x = \tanh y$, and so we have

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \operatorname{sech}^2 y \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\operatorname{sech}^2 y}.$$

Osborn's Rule suggests that

$$\operatorname{sech}^2 y = 1 - \tanh^2 y.$$

(We can and should verify this using the definitions.) Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1-\tanh^2 y} = \frac{1}{1-x^2}.$$

These three standard derivatives should be memorised; we will see their usefulness in the next lecture.

Recall (Ex 5.3.3 and Ex 5.5.3) that we solved integrals of the form

$$\int \sqrt{1-x^2}\,dx \qquad \text{or} \qquad \int \frac{1}{\sqrt{4-x^2}}\,dx$$

using the identity

$$\cos^2 u = 1 - \sin^2 u$$

to suggest the substitution $x = a \sin u$. From the identity

$$\cosh^2 u - \sinh^2 u = 1$$

we can now solve integrals of the form

$$\int \sqrt{x^2-1}\,dx \qquad \text{or} \qquad \int \frac{1}{\sqrt{4+x^2}}\,dx$$

by means of the substitution $x = a\cosh u$ or $x = a \sinh u$.

Example 6.8.3: Calculate

$$\int \sqrt{2x^2+4x-8}\,dx.$$

We have

$$2x^2+4x-8 = 2((x+1)^2-5)$$

and so

$$\int \sqrt{2x^2+4x-8}\,dx = \int \sqrt{2}\sqrt{(x+1)^2-(\sqrt{5})^2}\,dx.$$

Let $x+1 = \sqrt{5}\cosh u$ with $u \geq 0$. Then $\frac{\mathrm{d}x}{\mathrm{d}u} = \sqrt{5}\sinh u$ and

$$(x+1)^2 - (\sqrt{5})^2 = 5\sinh^2 u.$$

Now

$$\begin{aligned} \int \sqrt{2}\sqrt{5\sinh^2 u}\sqrt{5}\sinh u\,du &= 5\sqrt{2}\int \sinh^2 u\,du \\ &= \frac{5\sqrt{2}}{2}\int \cosh 2u - 1\,du \\ &= \frac{5\sqrt{2}}{2}\left[\frac{\sinh 2u}{2} - u\right] + C. \end{aligned}$$

But $\sinh 2u = 2\sinh u \cosh u = 2\cosh u\sqrt{\cosh^2 u - 1}$ (by our assumption on u) and so

$$\int \sqrt{2x^2+4x-8}\,dx$$

$$= \frac{5\sqrt{2}}{2}\left[\frac{x+1}{\sqrt{5}}\sqrt{\frac{(x+1)^2}{5}-1} - \cosh^{-1}\left(\frac{x+1}{\sqrt{5}}\right)\right] + C.$$

Example 6.8.4: Calculate

$$\int_0^1 \frac{1}{\sqrt{1+4x^2}}\,dx.$$

Let $2x = \sinh u$ so $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{2}\cosh u$ and

$$1 + 4x^2 = 1 + \sinh^2 u = \cosh^2 u.$$

Then

$$\int_0^{\sinh^{-1} 2} \frac{1}{\cosh u}\frac{\cosh u}{2}\,du = \left[\frac{u}{2}\right]_0^{\sinh^{-1} 2} = \frac{1}{2}\sinh^{-1} 2.$$

Generally we can quote (and hence should know)

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right)+C \qquad \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right)+C.$$

For integrals of the form

$$\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$$

we can now solve by completing the square.

Finally, we would like to have a more explicit formula for $\cosh^{-1} x$ and $\sinh^{-1} x$. As $\cosh x$ and $\sinh x$ are defined in terms of e^x, we might expect a formula involving ln.

Let $y = \sinh^{-1} x$, so $x = \sinh y$. Then

$$2x = e^y - e^{-y}.$$

Multiplying by e^y we see that

$$e^{2y} - 2xe^y - 1 = 0$$

and hence

$$(e^y - x)^2 - (x^2+1) = 0.$$

Solving for e^y we obtain

$$e^y = x \pm \sqrt{x^2+1}.$$

But $\sqrt{x^2+1} > x$ for all x, and $e^y \geq 0$ for all y. Hence

$$e^y = x + \sqrt{x^2+1}$$

and so

$$\sinh^{-1} x = y = \ln(x + \sqrt{x^2+1}).$$

In the same way we can show that

$$\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$$

(recall that we have only defined $\cosh^{-1} x$ for $x \geq 1$.) With these results we can now simplify our earlier examples.

Example 6.8.5: In Ex 6.6.2 we showed that

$$12\cosh^2 x + 7\sinh x = 24$$

had solutions $\sinh x = -\frac{4}{3}$ and $\sinh x = \frac{3}{4}$. By the above results we immediately obtain

$$x = \ln\left(-\frac{4}{3} + \sqrt{\frac{16}{9}+1}\right) = \ln\left(\frac{1}{3}\right) = -\ln(3)$$

and

$$x = \ln\left(\frac{3}{4} + \sqrt{\frac{9}{16}+1}\right) = \ln(2).$$