8. Set theory and logic

8.1 Sets and elements

A set is a collection of objects. The objects are called elements of the set. We write $x \in A$ for *x is an element of A*, and $x \notin A$ for *x is not an element of A*.

A set may be specified by listing its elements, e.g.

$$A = \{1, 4, 7, 11\}$$

or by stating a common property that defines the set, e.g.

the set of square numbers less than 100

or

$$\{x \in \mathbb{N} : x \text{ is prime}\}.$$

Some important sets are $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.

A set with a finite number of elements is called finite, otherwise it is called infinite.

If A and B are sets such that every element of A is also an element of B then we say that A is a subset of B, and write $A \subseteq B$.

If $A \subseteq B$ and $B \subseteq A$, i.e. A and B have exactly the same elements, we say that $A = B$.

If A is not a subset of B we write $A \not\subseteq B$.

Example 8.1.1:

(a)
$$\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}.$$

(b)
$$\{1, 2, 3, 4\} \subseteq \{1, 2, 3, 4\}.$$

(c)
$$\{1, 2, \{3, 4\}\} \not\subseteq \{1, 2, 3, 4\}.$$

(d)
$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

The set with no elements is called the empty set, denoted by $\emptyset$.

The set of all subsets of A is called the power set of A, written 2^A. If A is finite with n elements then 2^A has 2^n elements.

Example 8.1.2: The power set of

$$\{a, b, \{3, 4\}\}$$

is

$$\{\emptyset, \{a\}, \{b\}, \{\{3, 4\}\}, \{a, b\}, \{a, \{3, 4\}\}, \{b, \{3, 4\}\}, \{a, b, \{3, 4\}\}\}.$$

If $A \subseteq B$ and $A \neq B$ then we may write $A \subset B$ and call A a proper subset of B. In Example 7.2.1 the inclusions in (a) and (d) are proper, but not that in (b). The notation $\subseteq$ and $\subset$ is (deliberately) similar to $\leq$ and $<$ for real numbers.

We should really be a little more careful which collections of objects we call sets.

Example 8.1.3: (Russell's paradox)
Suppose there exists a set Ω of all sets. Let

$$R = \{A \in \Omega : A \notin A\}.$$

That is, R is the set of all sets which are not elements of themselves. Does R belong to R?

If $R \in R$ then by definition $R \notin R$!

If $R \notin R$ then by definition $R \in R$!

Thus R cannot exist.

Thus we cannot have all of the sets which we might imagine: in particular Ω and R *cannot exist* as sets.

To fix this we shall always assume that there is some fixed universe U of which all of our other sets are subsets. So then R and Ω are not in our universe U.

There are better (but more complicated) ways to avoid Russell's paradox, but this will suffice for our purposes.

8.2 Set operations

There are various ways to form new sets from old.

The union $A \cup B$ of two sets is the set of elements x such that $x \in A$ or $x \in B$ (including the possibility that x is in both). That is

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

The intersection $A \cap B$ of two sets is the set of elements x such that $x \in A$ and $x \in B$. That is

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

The complement of B in A, $A \backslash B$ is the set of elements $x \in A$ such that $x \notin B$. That is

$$A \backslash B = \{x : x \in A \text{ and } x \notin B\}.$$

We denote by B' the set

$$\{x : x \notin B\}.$$

Example 8.2.1: Let $A = \{1, 3, 5, 7\}$ and $B = \{2, 3, 4\}$.

$$A \cup B = \{1, 2, 3, 4, 5, 7\},$$

$$A \cap B = \{3\},$$

$$A \backslash B = \{1, 5, 7\}.$$

We can extend the notion of intersections and unions to collections of many sets. If $A_1, A_2, \ldots, A_n$ are sets then we define

$$\bigcup_{i=1}^{n} A_i = \{x : x \in A_i \text{ for some } i\}$$

and

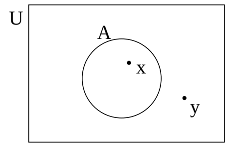
$$\bigcap_{i=1}^{n} A_i = \{x : x \in A_i \text{ for all } i\}.$$

We can even extend these definitions to infinite collections of sets.

8.3 Venn diagrams and membership tables

Venn diagrams are a means of visualising the relationship between various sets. They are only a guide, and no substitute for a proper proof!

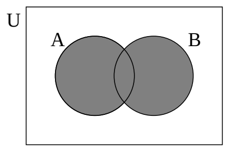
We represent the universe U by a rectangle, and each set A, B by a region in U. So



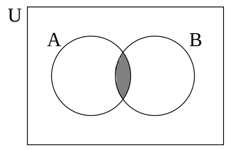
represents a set A, with $x \in A$ and $y \notin A$.

In this way we can visualise 'typical' configurations of sets. Often we shade the region of interest to us.

For example:

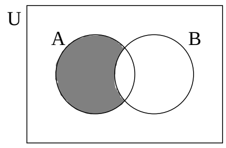


$A \cup B$

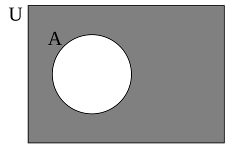


$A \cap B$

or:



$A \setminus B$

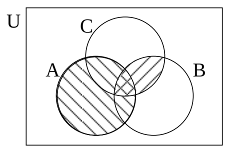


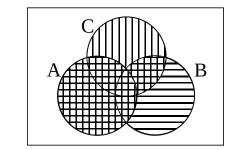
A'

Venn diagrams may suggest identities which can then be verified by other means.

Example 8.1.1: Illustrate the sets

$$A \cup (B \cap C) \quad \text{and} \quad (A \cup B) \cap (A \cup C).$$





The shaded area on the left is the same as the double shaded area on the right.

Once we have found a possible identity, we can use membership tables to verify it.

In a membership table we list all possible arrangements of elements in our sets, with 1 denoting membership of a set, and 0 denoting non-membership. For our basic operations we have

A	B	$A \cup B$	$A \cap B$	$A \setminus B$	A'
1	1	1	1	0	0
1	0	1	0	1	0
0	1	1	0	0	1
0	0	0	0	0	1

Two sets are identical if their entries in a membership table are identical.

Example 8.3.2: Prove that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Columns 5 and 8 are identical, so the result is proved.

8.4 Finite sets

If A is finite, let $n(A)$ denote the number of elements in A. We have

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

and

$$\begin{aligned} n(A \cup B \cup C) &= n(A) + n(B) + n(C) \\ &\quad - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C). \end{aligned}$$

These are examples of the inclusion-exclusion principle.

Example 8.4.1: In a survey of the popularity of 3 supermarkets A, B, and C, 100 people were interviewed.

30 used only A, 22 used only B, and 18 used only C.

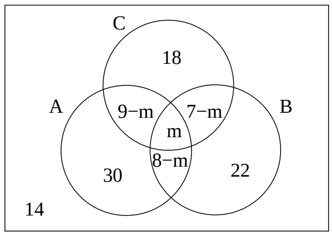
8 used A and B, 9 used A and C, and 7 used B and C.

14 used none of the three.

The survey was poorly designed, so it is not known how many of the people who shopped in a pair of supermarkets also shopped in the third.

How many people used all three supermarkets, and how many used A and C but not B?

Let m denote the number of people using all three supermarkets. We can illustrate the information graphically as follows:



We have

$$n(A \cup B \cup C) = 100 - 14 = 86.$$

Also,

$$86 = 30 + 22 + 18 + (8 - m) + (7 - m) + (9 - m) + m = 94 - 2m.$$

Therefore

$$n(A \cap B \cap C) = m = 4$$

and

$$n((A \cap C) \backslash B) = 9 - m = 5.$$

8.5 Propositional logic

A proposition is a statement which is either true (T) or false (F).

Example 8.5.1: (a) *"8 is an even number"* is a true proposition.
(b) *"The earth is flat"* is a false proposition.
(c) *"Stop writing!"* is not a proposition
(d) *"Are you hungry?"* is not a proposition.

We can make new propositions from old in various ways. We will describe these using a truth table (similar to a membership table).

The negation of a proposition p, denoted $\neg p$, is the proposition given by

p	$\neg p$
T	F
F	T

So the negation of 8.5.1(a) is *"8 is not an even number"*.

Given propositions p and q, we define the conjunction $p \wedge q$ ("p and q") by

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

The conjunction of 8.5.1(a) and (b) is *"8 is an even number and the earth is flat"*.

Given propositions p and q, we define the disjunction $p \vee q$ ("p or q") by

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The disjunction of 8.5.1(a) and (b) is *"8 is an even number or the earth is flat"*.

Given propositions p and q, we define the conditional $p \rightarrow q$ ("if p then q") by

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

If p equals 8.5.1(a) and q equals 8.5.1(b) then $p \rightarrow q$ is *"If 8 is an even number then the earth is flat"*.

Note that $p \to q$ is true whenever p is false. For example, if p is *"I win the election"* and q is *"I will give you a million pounds"* then $p \to q$ is *"If I win the election then I will give you a million pounds"*. If I lose the election you would not accuse me of lying if I had promised this (regardless of whether or not I give you any money)!

We say two propositions are equivalent if they have the same entries in a truth table. We will give some of the most useful examples of equivalent propositions.

The first two examples are known as De Morgan's Laws.

Example 8.5.2:

$$\neg(p \wedge q) = (\neg p) \vee (\neg q).$$

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$(\neg q)$	$(\neg p) \vee (\neg q)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

The fourth and seventh columns are equal, so the propositions are equivalent.

Example 8.5.3:

$$\neg(p \vee q) = (\neg p) \wedge (\neg q).$$

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$(\neg q)$	$(\neg p) \wedge (\neg q)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

The fourth and seventh columns are equal, so the propositions are equivalent.

Example 8.5.4:

$$\neg(p \to q) = p \wedge (\neg q).$$

p	q	$p \to q$	$\neg(p \to q)$	$(\neg q)$	$p \wedge (\neg q)$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

The fourth and sixth columns are equal, so the propositions are equivalent.

A proposition which is always true is called a tautology; one which is always false is called a contradiction.

Example 8.5.5:

$$p \vee (\neg p)$$

is a tautology.

p	$\neg p$	$p \vee (\neg p)$
T	F	T
F	T	T

There are two variants of $p \to q$ to which we give names. We call $q \to p$ the converse of $p \to q$, and $(\neg q) \to (\neg p)$ the contrapositive.

Note that a conditional is not equivalent to its converse. For example

"If X has robbed a bank then X is a criminal"

is not equivalent to

"If X is a criminal then X has robbed a bank"

(e.g. as X might be a murderer). However

Example 8.5.6: A conditional is equivalent to its contrapositive:

$$p \to q = (\neg q) \to (\neg p).$$

To see this we consider the truth table

p	q	$p \to q$	$\neg q$	$\neg p$	$(\neg q) \to (\neg p)$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

For example *"If it is raining then I need an umbrella"* is equivalent to *"If I do not need an umbrella then it is not raining"*.

Finally, we write $p \leftrightarrow q$ for $(p \to q) \wedge (q \to p)$.

8.6 Predicate logic

So far we have looked at propositional logic. There is a more general kind of logic called predicate logic.

A predicate is either a proposition as before, or it is a statement of the form

$$p(x_1, x_2, \ldots, x_n)$$

where each x_i is a variable coming from some set D_i, such that for each choice of $x_1, \ldots, x_n$ the statement $p(x_1, \ldots, x_n)$ is a proposition. This is a little bit complicated, so we will consider some examples.

Example 8.6.1: (i) Let $p(x)$ be *"x is greater than 3"*, where $x \in \mathbb{Z}$.
Then $p(2)$ is false, but $p(4)$ is true.
(ii) Let $q(x, y)$ be "$x^2 - y^2 < 0$" where $x \in \mathbb{Z}$ and $y \in \mathbb{R}$.
Then $q(1, 2)$ is "$1 - 4 < 0$" which is true, while $q(4, \sqrt{3})$ is "$16 - 3 < 0$" which is false.

Thus a predicate can be regarded as a function which for every choice of variables is either true or false.

We can make new predicates from old using $\vee$, $\wedge$, $\neg$, and $\rightarrow$.

Example 8.6.2: (i) Let $p(x)$ be "$x > 2$", where $x \in \mathbb{R}$ and $q(x)$ be "$x < 5$", where $x \in \mathbb{R}$. Then $p(x) \wedge q(x)$ is

"$x > 2$ and $x < 5$" with $x \in \mathbb{R}$.

This is true for $x = 3$ and false for $x = 1$.
(ii) Let p and q be as in (i). Then $p(x) \rightarrow q(x)$ is

"if $x > 2$ then $x < 5$" with $x \in \mathbb{R}$.

If $x = 3$ then this is true, and if $x = 6$ then this is false.

Note that when we write e.g. $p(x) \wedge q(x)$ we mean the same value for x in p and in q. If we write $p(x) \wedge q(y)$ then we can choose different values for x and y (although they can still be the same if we so choose).

Predicates are a useful way to write down certain statements, but they become much more useful in mathematics when combined with quantifiers. There are two quantifiers which we shall use.

The universal quantifier, written $\forall$, corresponds to the English phrase *"for all"* or *"for each"*. If $p(x)$ is a predicate with $x \in D$, then

$$(\forall x)(p(x))$$

means *"for all $x \in D$, $p(x)$ is true"*.

The existential quantifier, written $\exists$, corresponds to the English phrase *"there exists"* or *"for some"*. If $p(x)$ is a predicate with $x \in D$, then

$$(\exists x)(p(x))$$

means *"there exists $x \in D$ such that $p(x)$ is true"*.

Example 8.6.3: Let $h(x)$ mean *"x is happy"*, where x ranges over the set of mathematicians. Then

$(\forall x)(h(x))$ means *"all mathematicians are happy"*,
$(\exists x)(h(x))$ means *"there exists a happy mathematician"*.

Suppose that $a(x)$ means *"x is an algebraist"*, where x runs over the set of mathematicians. Then $(\forall x)(a(x) \longrightarrow h(x))$ means

"for all mathematicians x, if x is an algebraist then x is happy",

or put more simply, *"all algebraists are happy"*.

By combining quantifiers, we can express quite complicated ideas.

Example 8.6.4: Let $p(x, y)$ mean *"$x - y$ is an integer"*, where x and y are real numbers. Then

- $(\forall x)(\forall y)(p(x, y))$ means *"for all real numbers x and y, $x - y$ is an integer"* which is false (e.g take $x = 1$ and $y = 0.5$).
- $(\exists x)(\exists y)(p(x, y))$ means *"there exist real numbers x and y such that $x - y$ is an integer"* which is true (e.g take $x = 1$ and $y = 0$).
- $(\forall x)(\exists y)(p(x, y))$ means *"for all real numbers x there exists a real number y such that $x - y$ is an integer"* which is true (e.g take $y = x - 1$).
- $(\exists y)(\forall x)(p(x, y))$ means *"there exists a real number y such that for all real numbers x we have that $x - y$ is an integer"* which is false (e.g. we could take $x = y - 0.5$).

Note in particular the final pair of examples — the order in which we write quantifiers is very important!

Let us see how to write down some common mathematical phrases using quantifiers. Let $p(x)$ mean *"x is an A"* and $q(x)$ mean *"x is a B"*, where in each case x runs over some set D. Then *"Every A is a B"* can be written

$$(\forall x)(p(x) \longrightarrow q(x)).$$

"No A is a B" can be written

$$(\forall x)(p(x) \longrightarrow (\neg q(x))).$$

"Some As are Bs" can be written

$$(\exists x)(p(x) \wedge q(x)).$$

"Some A is not a B" can be written

$$(\exists x)(p(x) \wedge (\neg q(x))).$$

We have seen how to negate the various operations of proposition logic (for "and" and "or" these identities were known as De Morgan's laws). We would also like to be able to negate a statement involving quantifiers.

Suppose that $p(x)$ is a predicate where x ranges over a set D. We have

- $(\forall x)(p(x))$ meaning *"all x have property p"*.
- $(\exists x)(\neg p(x))$ meaning *"some x does not have property p"*.
- $(\exists x)(p(x))$ meaning *"some x has property p"*.
- $(\forall x)(\neg p(x))$ meaning *"no x has property p"*.

Now (2) is the denial of (1), and so

$$\neg((\forall x)(p(x))) \text{ is the same as } (\exists x)(\neg p(x)).$$

Also (4) is the denial of (3), and so

$$\neg((\exists x)(p(x))) \text{ is the same as } (\forall x)(\neg p(x)).$$

Example 8.6.5: (i) If $p(x)$ means *"x is positive"* where $x \in \mathbb{Z}$, then $\neg((\forall x)(p(x)))$ means

"not all integers are positive",

which is the same as saying

"there exists an integer which is not positive",

i.e. $(\exists x)(\neg p(x))$.

(ii) Similarly, if $q(x)$ means *"x is even"* where x runs over the set of prime numbers then $\neg((\exists x)(q(x)))$ means

"there does not exist an even prime number",

which is the same as saying

"all prime numbers are odd",

i.e. $(\forall x)(\neg q(x))$.