Representation theory of finite dimensional algebras

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Contents

Introduction

This course will provide a basic introduction to the representation theory of algebras, concentrating mainly on the finite dimensional case. Representation theory is concerned with the study of how various algebraic objects act on vector spaces, in a manner which respects the original algebraic structure. Finite dimensional algebras, while of interest in their own right, provide a (relatively) elementary setting in which to develop some of the basic language, while still exhibiting most of the key features that can arise.

It is common for a first course in representation theory to concentrate on the character theory of finite groups over the complex numbers. This has a number of advantages, not least that characters are much easier to construct than the corresponding representations. However, the theory is rather unrepresentative in certain crucial respects.

Most important of these is that such representations are always semisimple. This means that it is enough to classify those representations which have no sub-representations, which (for a given group) is a finite number. All other representations can then be constructed from these via direct sums. In general one cannot hope to construct all representations of an algebra. In fact, we will see that such an aim is provably impossible to achieve (except in certain special cases). Instead we will develop various tools to analyse representations in general.

In this course we will focus not on group algebras (although these will play a role), but rather on certain algebras associated to quivers. These have many advantages (even over group algebras) in terms of ease of computation and of constructing examples, but are rich enough to give a better flavour of general aspects of representation theory. Indeed, it will turn out that to understand the representation theory of *any* finite dimensional algebra over an algebraically closed field it is enough to understand the representation theory associated to quivers and quotients of quivers. This includes group algebras as a special case (and over any algebraically closed field, not just the complex numbers).

In Chapter 1 we will begin with various basic definitions and examples. First we will look at algebras and modules, and then at quivers and their representations. We will then see that the quiver setting gives rise to examples in the algebra setting.

Chapter 2 covers the core classical representation theory of algebras. We begin with an analysis of the relation between simple representations and representations in general, and then consider for which algebras we can reduce to the study of simples alone. Such algebras are called semisimple, and the Artin-Wedderburn Theorem will give a complete classification in this case.

If an algebra is not semisimple, then the Jacobson radical of the algebra can be regarded as a measure of its non-semisimplicity. We will develop the basic properties of this. The Krull-Schmidt Theorem then tells us that it is enough to determine the indecomposable modules (a class which contains the simple modules but is in general much larger).

In non-semisimple settings projective and injective modules play a key role. In this course we will only be able to touch on the basic definitions, and will say nothing of the vital role they play in cohomology. This is in part because we will not have the time to develop the necessary background in category theory which is an important part of modern day algebra.

Chapter 3 gives the basic definitions of projective and injectives, before going on to a study of the role of idempotents in representation theory. Using these is what allows us to reduce to the study of quivers, although we will only give an outline of the reduction method here. We then show how simple, projective, and injective modules can be easily constructed for quivers.

Chapter 4 is an introduction to some of the basic notions of category theory that play a role in representation theory. We introduce categories and functors, and explain the notions of natural transformations and of equivalence betweeen categories. We briefly sketch the statement of the Freyd-Mitchell embedding theorem, which indicates the fundamental nature of module categories.

Chapter 5 introduces the notion of representation type. This is a measure of how hard it is to fully understand the representation theory of an algebra. The fundamental theorem of Drozd says that every algebra falls into one of three types; the first two being (in principle) completely understandable, while the third is provably impossible to fully understand.

We will sketch how the classification by type can be carried out in two special cases: group algebras, and for representations of quivers without relations. The latter case will allow us to introduce some more ideas from the representation theory of quivers which are used in the proof.

Finally in Chapter 6 we will indicate some further topics which the reader may wish to investigate.

Recommended reading

Due to the limited number of lectures available, the lectures will consist of an outline of the main theory together with some examples. These notes will fill in more of the details, but with only sketches of the proofs in places. Each Chapter ends with a brief selection of exercises for the reader. For a far more comprehensive treatment of this material (together with many more examples and exercises) the reader is recommended to look at [**ASS06**, Chapters I-III].

For simplicity, [**ASS06**] only considers algebras over algebraically closed fields. An excellent (if rapid) introduction which considers more general rings can be found in [**Ben91**, Chapters 1 and 4]. Other notes available on the web which cover similar material are [**Bru03**] and [**Bar06**]. Our exposition draws on all of these sources, as well as unpublished lecture notes of Erdmann. For the basic theory of categories we draw on [**Mac97**].

Two books which go into more advanced topics than this course (and which are not entirely suitable for the beginner) are [**ARS94**] and [**GR97**].