CHAPTER 1

Algebras and modules

In this course we will be interested in the representation theory of finite dimensional algebras defined over a field. We begin by recalling certain basic definitions concerning fields.

DEFINITION 1.0.1. *A field k is* algebraically closed *if every non-constant polynomial with coefficients in k has a root in k. A field has* characteristic p *if p is the smallest positive integer such that*

$$\sum_{i=1}^{p} 1 = 0.$$

If there is no such p then the field is said to have characteristic 0. *A field is* infinite *if it contains infinitely many elements.*

Henceforth k will denote some field.

1.1. Associative algebras

DEFINITION 1.1.1. *An* algebra over k, *or* k-algebra *is a k-vector space A with a bilinear map*

$$\begin{array}{ccc} A \times A & \longrightarrow & A \\ (x,y) & \longmapsto & xy. \end{array}$$

We say that the algebra is associative *if for all $x,y,z \in A$ we have*

$$x(yz) = (xy)z.$$

An algebra A is unital *if there exists an element $1 \in A$ such that $1x = x1 = x$ for all $x \in A$. Such an element is called the* identity *in A. (Note that such an element is necessarily unique.) We say that an algebra is* finite dimensional *if the underlying vector space is finite dimensional. An algebra A is* commutative *if $xy = yx$ for all $x,y \in A$.*

It is common to abuse terminology and take algebra to mean an associative unital algebra, and we will follow this convention. There are several important classes of non-associative algebras (for example Lie algebras) but we shall not consider them here. **Thus all algebras we consider will be associative and unital.**

EXAMPLE 1.1.2. *(a) Let $k[x_1,\ldots,x_n]$ denote the vector space of polynomials in the (commuting) variables $x_1,\ldots,x_n$. This is an infinite dimensional commutative algebra with multiplication given by the usual multiplication of polynomials, and identity given by the trivial polynomial* 1.

(b) Let $k\langle x_1,\ldots,x_n\rangle$ denote the vector space of polynomials in the non-commuting *variables $x_1,\ldots,x_n$. A general element is of the form $\sum_{i=1}^{n} \lambda_i w_i$ for some n where for each i, $\lambda_i \in k$ and*

$w_i = x_{i_1}^{a_1} x_{i_2}^{a_2} \dots x_{i_t}^{a_t}$ *for some* t*. Given two elements* $\sum_{i=1}^n \lambda_i w_i$ *and* $\sum_{i=1}^m \lambda_i' w_i'$ *the product is defined to be the element*

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \lambda_i' w_i w_j'$$

where $w_i w_j$ *denotes the element obtained from* w_i *and* w_j *by concatenation. This is an infinite dimensional associative algebra with identity given by the trivial polynomial* 1*. If* $n > 1$ *then the algebra is non-commutative.*

(c) Given a group G*, we denote by* kG *the* group algebra *obtained by considering the vector space of formal linear combinations of group elements. Given two elements* $\sum_{i=1}^n \lambda_i g_i$ *and* $\sum_{i=1}^m \mu_i h_i$ *with* $\lambda_i, \mu_i \in k$ *and* $g_i, h_i \in G$ *we define the product to be the element*

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j g_i h_j.$$

The identity element is the identity element $e \in G$ *regarded as an element of* kG*. The algebra* kG *is finite dimensional if and only if* G *is a finite group, and is commutative if and only if* G *is abelian.*

(d) The set $M_n(k)$ *of* $n \times n$ *matrices with entries in* k *is a finite dimensional algebra, the* matrix algebra, *with the usual matrix multiplication, and identity element the matrix* I*. If* $n > 1$ *it is non-commutative. Equivalently, let* V *be an* n*-dimensional* k*-vector space, and consider the* endomorphism algebra

$$\mathrm{End}_k(V) = \{ f : V \longrightarrow V \mid f \text{ is } k\text{-linear} \}.$$

This is an algebra with multiplication given by composition of functions. Fixing a basis for V *the elements of* $\mathrm{End}_k(V)$ *can be written in terms of matrices with respect to this basis, and in this way we can identify* $\mathrm{End}_k(V)$ *with* $M_n(k)$*.*

(e) If A *is an algebra then so is* A^{op}*, the* opposite algebra, *which equals* A *as a vector space, but with multiplication map* $(x,y) \longmapsto yx$*.*

As usual in Algebra, we are not just interested in objects (in this case algebras), but also in functions between them which respect the underlying structures.

DEFINITION 1.1.3. *A* homomorphism *between* k*-algebras* A *and* B *is a linear map* $\phi : A \longrightarrow B$ *such that* $\phi(1) = 1$ *and* $\phi(xy) = \phi(x)\phi(y)$ *for all* $x, y \in A$*. This is an* isomorphism *precisely when the linear map is a bijection.*

DEFINITION 1.1.4. *Given an algebra* A*, a* subalgebra *of* A *is a subspace* S *of* A *containing* 1*, such that for all* $x, y \in S$ *we have* $xy \in S$*. A* left (respectively right) ideal *in* A *is a subspace* I *of* A *such that for all* $x \in I$ *and* $a \in A$ *we have* $ax \in I$ *(respectively* $xa \in I$*). If* I *is a left and a right ideal then we say that* I *is an* ideal *in* A*.*

EXAMPLE 1.1.5. *(a) If* H *is a subgroup of a group* G*, then* kH *is a subalgebra of* kG*.*

(b) Given two algebras A *and* B*, and a homomorphism* $\phi : A \longrightarrow B$*, the set* $\mathrm{im}(\phi)$ *is a subalgebra of* B*, while* $\ker(\phi)$ *is an ideal in* A*.*

Idempotents play a crucial role in the analysis of algebras.

DEFINITION 1.1.6. *An element* $e \in A$ *is an* idempotent *if* $e^2 = e$*. Two idempotents* e_1 *and* e_2 *in* A *are* orthogonal *if*

$$e_1 e_2 = e_2 e_1 = 0.$$

An idempotent e is called primitive *if it cannot be written in the form $e = e_1 + e_2$ where e_1 and e_2 are non-zero orthogonal idempotents. An idempotent e is* central *if $ea = ae$ for all $a \in A$.*

1.2. Modules

Representation theory is concerned with the study of the way in which certain algebraic objects (in our case, algebras) act on vector spaces. There are two ways to express this concept; in terms of representations or (in more modern language) in terms of modules.

DEFINITION 1.2.1. *Given an algebra A over k, a* representation *of A is an algebra homomorphism*

$$\phi : A \longrightarrow \mathrm{End}_k(M)$$

for some vector space M. A left A-module *is a k-vector space M together with a bilinear map $A \times M \longrightarrow M$, which we will denote by $(a,m) \longmapsto am$, such that for all $m \in M$ and $x,y \in A$ we have $1m = m$ and $(xy)m = x(ym)$. Similarly, a* right A-module *is a k-vector space M and a bilinear map $\phi : M \times A \longrightarrow M$ such that $m1 = m$ and $m(xy) = (mx)y$ for all $m \in M$ and $x,y \in A$. We will adopt the convention that all modules are left modules unless stated otherwise.*

DEFINITION 1.2.2. *An A-module is* finite dimensional *if it is finite dimensional as a vector space. An A-module M is* generated *by a set $\{m_1 : i \in I\}$ (where I is some index set) if every element m of M can be written in the form*

$$m = \sum_{i \in I} a_i m_i$$

for some $a_i \in A$. We say that M is finitely generated *if it is generated by a finite set of elements. If A is a finite dimensional algebra then M is finitely generated if and only if M is finite dimensional.*

LEMMA 1.2.3. *(a) There is a natural equivalence between left (respectively right) A-modules and right (respectively left) A^{op}-modules.*
(b) There is a natural equivalence between representations of A and left A-modules.

PROOF. We give the correspondence in each case; details are left to the reader. Given a left module M for A with bilinear map $\phi : A \times M \longrightarrow M$, define a right A^{op}-module structure on M via the map $\phi' : M \times A \longrightarrow M$ given by $\phi'(m,x) = \phi(x,m)$. It is easy to verify that ϕ is an A^{op}-homomorphism.

Given a representation $\phi : A \longrightarrow \mathrm{End}_k(M)$ of A we define an A-module structure on M by setting

$$am = \phi(a)(m)$$

for all $a \in A$ and $m \in M$. Conversely, given an A-module M, the map $M \longrightarrow M$ given by $m \longmapsto rm$ is linear, and gives the desired representation $\phi : A \longrightarrow \mathrm{End}_k(M)$. □

DEFINITION 1.2.4. *A* homomorphism *between A-modules M and N is a linear map $\phi : M \longrightarrow N$ such that $\phi(am) = a\phi(m)$ for all $a \in A$ and $m \in M$. This is an* isomorphism *precisely when the linear map is a bijection.*

DEFINITION 1.2.5. *Given an A-module M, a* submodule *of M is a subspace N of M such that for all $n \in N$ and $a \in A$ we have $an \in N$. (Note that N is an A-module in its own right.) The quotient space*

$$M/N = \{m + N : m \in M\}$$

(under the relation $m+N=m'+N$ if and only if $m-m'\in N$) has an A-module structure given by $a(m+N)=am+N$, and is called the quotient *of M by N.*

EXAMPLE 1.2.6. *(a) The algebra A is a (left or right) A-module, with respect to the usual multiplication map on A. If I is a left ideal of A then I is a submodule of the left module A.*

(b) If $A=k$ then A-modules are just k-vector spaces.

(c) If $A=k[x_1,\ldots,x_n]$ then an A-module is a k-vector space M together with commuting linear transformations $\alpha_i : M \longrightarrow M$ (where α_i describes the action of x_i).

(d) Every A-module M has M and the empty vector space 0 as submodules.

LEMMA 1.2.7 (Isomorphism Theorem). *If M and N are A-modules and $\phi : M \longrightarrow N$ is a homomorphism of A-modules then*

$$\mathrm{im}(\phi)\cong M/\ker(\phi)$$

as A-modules.

PROOF. Copy the proof for linear maps between vector spaces, noting that the additional structure of a module is preserved. □

DEFINITION 1.2.8. *If an A-module M has submodules L and N such that $M=L\oplus N$ as a vector space then we say that M is the* direct sum *of L and N. A module M is* indecomposable *if it is not the direct sum of two non-zero submodules (and is* decomposable *otherwise). A module M is* simple *(or* irreducible*) if M has no submodules except M and 0.*

For vector spaces, the notions of indecomposability and irreducibility coincide. However, this is not the case for modules in general.

EXAMPLE 1.2.9. *Let C_2 denote the cyclic group with elements $\{1,g\}$, and consider the two-dimensional kC_2-module M with basis $\{m_1,m_2\}$ where $gm_1=m_2$ and $gm_2=m_1$. If $M=N_1\oplus N_2$ with N_1 and N_2 non-zero then each N_i is the span of a vector of the form $\lambda_1m_1+\lambda_2m_2$ for some $\lambda_1,\lambda_2\in k$. Applying g we deduce that $\lambda_1=\pm\lambda_2$, and hence N_i must be the span of m_1-m_2 or m_1+m_2. But $N_1=N_2$ if k has characteristic 2, which contradicts our assumption. Thus M is never irreducible, but is indecomposable if and only if the characteristic of k is 2. We will see that this example generalises to arbitrary group algebras when we consider Maschke's Theorem.*

There is a close relationship between the representation theory of A and A^{op}.

DEFINITION 1.2.10. *Let M be a finite dimensional (left) A-module. Then the* dual module *M^* is the dual vector space $\mathrm{Hom}_k(M,k)$ with a right A-module action given by $(\phi a)(m)=\phi(am)$ for all $a\in A$, $m\in M$ and $\phi\in\mathrm{Hom}_k(M,k)$. By Lemma 1.2.3 this gives M^* the structure of a left A^{op}-module.*

Taking the dual of an A^{op}-module gives an A-module, and it is easy to verify (as for vector spaces) that

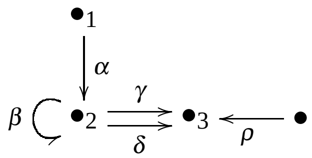
LEMMA 1.2.11. *For any finite dimensional A-module M we have $M^{**}\cong M$.*

1.3. Quivers

DEFINITION 1.3.1. *A* quiver Q *is a directed graph. We will denote the set of vertices by* Q_0*, and the set of edges (which we call* arrows*) by* Q_1*. If* Q_0 *and* Q_1 *are both finite then* Q *is a* finite quiver*. The* underlying graph $\overline{Q}$ *of a quiver* Q *is the graph obtained from* Q *by forgetting all orientations of edges.*

A path of length n *in* Q *is a sequence* $p = \alpha_1\alpha_2\dots\alpha_n$ *where each* α_i *is an arrow and* α_i *starts at the vertex where* α_{i+1} *ends. For each vertex* i*, there is a path of length* 0*, which we denote by* ε_i*. A quiver is* acyclic *if the only paths which start and end at the same vertex have length* 0*, and* connected *if* $\overline{Q}$ *is a connected graph.*

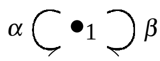
EXAMPLE 1.3.2. *(a) For the quiver* Q *given by*



the set of paths of length greater than 1 *is given by*

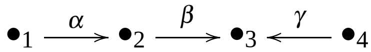
$$\{\beta^{n+2}, \beta^{n+1}\alpha, \gamma\beta^{n+1}, \delta\beta^{n+1}, \gamma\beta^n\alpha, \delta\beta^n\alpha : n \geq 0\}.$$

(b) For the quiver Q *given by*



the set of paths corresponds to words in α *and* β *(along with the trivial word).*

(c) For the quiver Q *given by*



the set of paths is

$$\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \alpha, \beta, \gamma, \beta\alpha\}.$$

We would like to associate an algebra to a quiver; however, we need to take a little care.

DEFINITION 1.3.3. *The* path algebra kQ *of a quiver* Q *is the* k*-vector space with basis the set of paths in* Q*. Multiplication is via concatenation of paths: if* $p = \alpha_1\alpha_2\dots\alpha_n$ *and* $q = \beta_1\beta_2\dots\beta_m$ *then*

$$pq = \alpha_1\alpha_2\dots\alpha_n\beta_1\beta_2\dots\beta_m$$

if α_n *starts at the vertex where* β_1 *ends, and is* 0 *otherwise.*

We have not yet checked that the above definition does in fact define an algebra structure on kQ.

LEMMA 1.3.4. *Let* Q *be a quiver. Then* kQ *is an associative algebra. Further* kQ *has an identity element if and only if* Q_0 *is finite, and is finite dimensional if and only if* Q *is finite and acyclic.*

PROOF. The associativity of multiplication in kQ is straightforward. Next note that the elements ε_i satisfy

$$\varepsilon_i\varepsilon_j = \delta_{ij}\varepsilon_i$$

and hence form a set of orthogonal idempotents. Further, for any path $p \in kQ$ we have $\varepsilon_i p = p$ if p ends at vertex i and 0 otherwise. Hence if Q_0 is finite then

$$\sum_{i\in Q_0} \varepsilon_i p = p.$$

Similarly

$$\sum_{i\in Q_0} p\varepsilon_i = p$$

and hence

$$1 = \sum_{i\in Q_0} \varepsilon_i$$

is the unit in kQ.

Conversely, suppose that Q_0 is infinite and $1 \in kQ$. Then $1 = \sum \lambda_i p_i$ for some (finite) set of paths p_i and scalars λ_i. Pick a vertex j such that for all i the path p_i does not end at j. Then $\varepsilon_j 1 = 0$, which gives a contradiction.

Finally, if Q_0 or Q_1 is not finite then kQ is clearly not finite dimensional. Given a finite set of vertices with finitely many edges, there are only finitely many paths between them unless the quiver contains a cycle. □

EXAMPLE 1.3.5. *Each of the quivers in Example 1.3.2 is finite, and so the corresponding kQ contains a unit. However, the path algebras corresponding to 1.3.2(a) and 1.3.2(b) are not finite dimensional. Indeed, it is easy to see that the path algebra for (b) is isomorphic to* $k\langle x,y\rangle$*, under the map taking* α *to* x *and* β *to* y*. The path algebra for 1.3.2(c) is an* 8*-dimensional algebra.*

Because of Lemma 1.3.4 we will only consider finite quivers Q, so that the corresponding path algebras are unital.

DEFINITION 1.3.6. *Given a finite quiver Q, the ideal* R_Q *of kQ generated by the arrows in Q is called the* arrow ideal *of kQ. Then* R_Q^m *is the ideal generated by all paths of length m in Q. An ideal I in kQ is called* admissible *if there exists* $m \geq 2$ *such that*

$$R_Q^m \subseteq I \subseteq R_Q^2.$$

If I is admissible then (Q,I) *is called a* bound quiver*, and* kQ/I *is a* bound quiver algebra.

Note that if Q is finite and acyclic then any ideal contained in R_Q^2 is admissible, as $R_Q^m = 0$ if m is greater than the maximal path length in Q.

EXAMPLE 1.3.7. *Let Q be as in Example 1.3.2(b), and let* $I = \langle\beta\alpha,\beta^2\rangle$*. This is not an admissible ideal in kQ as it does not contain* α^m *for any* $m \geq 1$*, and so does not contain* R_Q^m *for any* $m \geq 2$.

PROPOSITION 1.3.8. *Let Q be a finite quiver with admissible ideal I in kQ. Then* kQ/I *is finite dimensional.*

PROOF. As I is admissible there exists $m \geq 2$ such that $R_Q^m \subseteq I$. Hence there is a surjective algebra homomorphism from kQ/R_Q^m onto kQ/I. But the former algebra is clearly finite dimensional as there are only finitely many paths of length less than m. □

DEFINITION 1.3.9. *A* relation *in kQ is a finite linear combination of paths of length at least two in Q such that all paths have the same start vertex and the same end vertex. If* $\{\rho_j : j \in J\}$ *is a set of relations in kQ such that the ideal generated by the set is admissible then we say that kQ is* bound by the relations.

EXAMPLE 1.3.10. *Consider the quiver in Example 1.3.2(a) and the relations*

$$\{\gamma\beta^2\alpha - \delta\alpha, \gamma\beta + \delta\beta, \beta^5\}.$$

Any path of length at least 7 *must contain* β^5*, and so Q is bound by this set of relations.*

In fact the above example generalises: it can be shown that any ideal I in R_Q^2 is admissible if it contains each cycle in Q to some power. Further, we have

PROPOSITION 1.3.11. *Let Q be a finite quiver. Every admissible ideal in kQ is generated by a finite sequence of relations in kQ.*

PROOF. (Sketch) It is easy to check that every admissible ideal I is finitely generated by some set $\{a_1, \ldots, a_n\}$ (as R_Q^m and I/R_Q^m are finitely generated). However, in general a set of generators for I will not be a set of relations, as the paths in each a_i may not all have the same start vertex and end vertex. However, the non-zero elements in the set

$$\{\varepsilon_x a_i \varepsilon_y : 1 \leq i \leq n, x, y \in Q_0\}$$

are all relations, and this set generates I. □

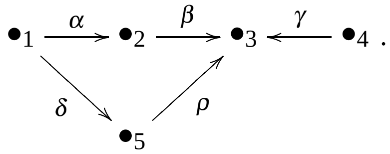
1.4. Representations of quivers

DEFINITION 1.4.1. *Let Q be a finite quiver. A* representation *M of Q over k is a collection of k-vector spaces* $\{M_a : a \in Q_0\}$ *together with a linear map* $\phi_\alpha : M_a \longrightarrow M_b$ *for each arrow* $\alpha : a \longrightarrow b$ *in* Q_1*. The representation M is* finite dimensional *if all the* M_a *are finite dimensional.*

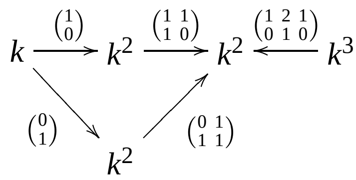
DEFINITION 1.4.2. *Given two representations M and* M' *of a finite quiver Q, a* homomorphism from M to N *is a collection of linear maps* $f_i : M_i \longrightarrow M_i'$ *such that for each arrow* $\alpha : i \longrightarrow j$ *we have* $\phi'_\alpha f_i = f_j \phi_\alpha$.

When giving examples of representations of quivers we will usually fix bases of each of the vector spaces, and represent the maps between them by matrices with respect to column vectors in these bases.

EXAMPLE 1.4.3. *Consider the quiver*



This has a representation



Notice how easy it was to give a representation: there are no compatibility relations to be checked (apart from that the linear maps go between the appropriate dimension) so examples can be easily generated for any path algebra. This is very different from writing down explicit modules for an algebra (in general).

Definition 1.4.1 looks rather different from that for an algebra. However, the next lemma shows that representations of Q correspond to kQ-modules in a natural way.

LEMMA 1.4.4. *Let M be a representation of a finite acyclic quiver Q. Consider the vector space*

$$M' = \bigoplus_{a \in Q_0} M_a.$$

This can be given the structure of a kQ-module by defining for each $\alpha : i \longrightarrow j$ a map $\phi'_\alpha : M \longrightarrow M$ by

$$\phi'_\alpha(m_1, \ldots, m_n) = (0, \ldots, 0, \phi_\alpha(m_i), 0, \ldots 0)$$

where the non-zero entry is in position j, and for each $i \in Q_0$ a map $\varepsilon_i : M \longrightarrow M$ by

$$\varepsilon_i(m_1, \ldots, m_n) = (0, \ldots, 0, m_i, 0, \ldots, 0)$$

where the non-zero entry is in position i. Conversely, suppose that N is a kQ-module. Then we obtain a representation of Q by setting $N_a = \varepsilon_a N$ and defining ϕ_α for $\alpha : a \longrightarrow b$ to be the restriction of the action of $\alpha \in kQ$ to N_a.

PROOF. Checking that the above definitions give a kQ-module and a representation of Q respectively is routine. □

We also need the notion of a representation of a bound quiver. Note that we do not need to assume that Q is acyclic here, as admissible ideals guarantee that the associated quotient algebra is finite dimensional.

DEFINITION 1.4.5. *Given a path $p = \alpha_1\alpha_2 \ldots \alpha_n$ in a finite quiver Q from a to b and a representation M of Q we define the linear map ϕ_p from M_a to M_b by*

$$\phi_p = \phi_{\alpha_n}\phi_{\alpha_{n-1}} \cdots \phi_{\alpha_1}.$$

If ρ is a linear combination of paths p_i with the same start vertex and the same end vertex then ϕ_ρ is defined to be the corresponding linear combination of the ϕ_{p_i}. Given an admissible ideal I in kQ we say that M is bound by *I if $\phi_\rho = 0$ for all relations $\rho \in I$.*

EXAMPLE 1.4.6. *Consider the representation in Example 1.4.3. Let $p = \beta\alpha$ and $q = \rho\delta$. Then*

$$\phi_p = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \phi_q = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and so this representation is bound by the ideal $\langle \beta\alpha - \rho\delta \rangle$.

It is easy to verify that the correspondence between representation of finite acyclic Q and kQ-modules given in Lemma 1.4.4 extends to a correspondence between representations of finite Q bound by I and kQ/I-modules.

The language of categories and functors is a very powerful one, and many results in representation theory are best stated in this way. Roughly, a category is a collection of *objects* (e.g. kQ-modules) and *morphisms* (e.g. kQ-homomorphisms), and the idea is to study the category as a whole rather than just the objects or morphisms separately. A *functor* is then a map from one category to another which transports both objects and morphisms in a suitably compatible way. In this language the above result relating bound representations of Q and kQ/I-modules gives an equivalence between the corresponding categories. We will make this more precise in a later chapter.

1.5. Exercises

(1) Suppose that I is an ideal in an algebra A.

  (a) Show that A/I has an algebra structure such that there is a surjective homomorphism from A to A/I.
  (b) Suppose that A is an algebra with ideal I, and that M is an A/I-module. Show that M can be given the structure of an A-module.
  (c) If M is an A-module, what condition must it satisfy to be an A/I-module?

(2) Suppose that $(P, \leq)$ is a partially ordered set of cardinality n, and define kP to be the subset of $M_n(k)$ given by

$$kP = \{M = (m_{ij}) : m_{ij} = 0 \text{ if } \; i \not\leq j\}.$$

  (a) Show that kP is a subalgebra of $M_n(k)$ (this is called the *incidence algebra* of $(P, \leq)$).
  (b) Show that P can be identified with the set $\{1, \ldots n\}$ in such a way that kP can be identified with a subalgebra of the algebra $LT_n(k)$ of lower triangular matrices in $M_n(k)$.
  (c) Deduce that if Q is a finite acyclic quiver with at most one arrow between each pair of vertices, then kQ is a subalgebra of $LT_n(k)$ for some n.
  (d) Illustrate your last construction in the case of the quiver in Example 1.3.2(c).
  (e) Which quiver correspond to the whole of $LT_n(k)$?

(3) Suppose that Q is a quiver, and let Q^{op} be the quiver obtained by reversing all the arrows. Show that there is an isomorphism of algebras $k(Q^{op}) \cong (kQ)^{op}$.

(4) Suppose that G is a group. Show that $kG \cong (kG)^{op}$.

(5) Classify the simple modules for the cyclic group C_n over an algebraically closed field of characteristic $p \geq 0$.

(6) Suppose that $M = (M_a, \phi_a)$ is a representation of some finite quiver Q.

(a) Given vector spaces $N_a \leq M_a$, what conditions must be satisfied for (N_a, ϕ_a) to be a subrepresentation N of Q?
(b) Suppose that M is a representation of Q bound by an admissible ideal I. Show that the representation N is also bound by I.
(c) If Q has n vertices, give n non-isomorphic simple representations of kQ, and also of kQ/I. (Hint: what condition on the dimensions of the N_a guarantees the absence of a proper subrepresentation?)
(d) If Q is acyclic then we will see in Chapter 2 that these examples form a complete set of simple representations. However, it is also possible to show this directly. Suppose that M is a representation of an acyclic Q such that more than one M_a is non-zero. Show that M has a proper subrepresentation.
(e) Suppose that Q is finite but contains some cycle. Show that Q now has infinitely many non-isomorphic simple representations over $\mathbb{C}$.

(7) In this exercise we will classify the indecomposable representations of the quiver Q given by

$$\bullet_1 \xrightarrow{\alpha_1} \bullet_2 \xrightarrow{\alpha_2} \bullet_3 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_{n-2}} \bullet_{n-1} \xrightarrow{\alpha_{n-1}} \bullet_n \ .$$

Let $M = (M_i, \phi_i)$ be an indecomposable representation of Q.
(a) Show that if ϕ_i is not injective then $M_j = 0$ for $j > i$.
(b) Similarly show that if ϕ_i is not surjective then $M_j = 0$ for $j \leq i$.
(c) Deduce that M is isomorphic to a representation of the form

$$0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow k \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} k \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \ .$$

(d) Show that the $\frac{n(n+1)}{2}$ such modules are pairwise non-isomorphic.

We will see in Chapter 4 that this example is part of a more general picture.

(8) Let S_3 denote the symmetric group on three symbols. Decompose the group algebra $\mathbb{C}S_3$ into a direct sum of simple representations for S_3. (You may find it convenient to identify $\mathbb{C}S_3$ with a space of permutation matrices.)