CHAPTER 2

Semisimplicity and some basic structure theorems

In this chapter we will review some of the classical structure theorems for finite dimensional algebras. In most cases results will be stated with only a sketch of the proof. Henceforth we will restrict our attention to finite dimensional modules.

2.1. Simple modules and semisimplicity

Recall that a simple module is a module S such that the only submodules are S and 0. These form the building blocks out of which all other modules are made:

LEMMA 2.1.1. *If M is a finite dimensional A-module then there exists a sequence of submodules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that M_i/M_{i-1} is simple for each $1 \leq i \leq n$. Such a series is called a composition series *for M.*

PROOF. Proceed by induction on the dimension of M. If M is not simple, pick a submodule M_1 of minimal dimension, which is necessarily simple. Now $\dim(M/M_1) < \dim M$, and so the result follows by induction. □

Moreover, we have

THEOREM 2.1.2 (Jordan-Hölder). *Suppose that M has two composition series*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_m = M, \qquad 0 = N_0 \subset N_1 \subset \cdots \subset N_n = M.$$

Then $n = m$ and there exists a permutation σ of $\{1, \ldots n\}$ such that

$$M_i/M_{i+1} \cong N_{\sigma(i)}/N_{\sigma(i)+1}.$$

PROOF. The proof is similar to that for groups. □

Life would be (relatively) straightforward if every module was a direct sum of simple modules.

DEFINITION 2.1.3. *A module M is* semisimple *(or* completely reducible*) if it can be written as a direct sum of simple modules. An algebra A is* semisimple *if every finite dimensional A-module is semisimple.*

LEMMA 2.1.4. *If M is a finite dimensional A-module then the following are equivalent:*
(a) If N is a submodule of M then there exists L a submodule of M such that $M = L \oplus N$.
(b) M is semisimple.
(c) M is a (not necessarily direct) sum of simple submodules.

PROOF. (Sketch) Note that (a) implies (b) and (b) implies (c) are clear. For (c) implies (a) consider the set of submodules of A whose intersection with N is 0. Pick one such, L say, of maximal dimension; if $N \oplus L \neq M$ then there is some simple S in M not in $N \oplus L$. But this would imply that $S + L$ has intersection 0 with A, contradicting the maximality of L. □

LEMMA 2.1.5. *If M is a semisimple A-module then so is every submodule and quotient module of M.*

PROOF. (Sketch) If N is a submodule then $M = N \oplus L$ for some L by the preceding Lemma. But then $M/L \cong N$, and so it is enough to prove the result for quotient modules.

If M/L is a quotient module consider the projection homomorphism π from M to M/L. Write M as a sum of simple modules S_i and verify that $\pi(S)$ is either simple or 0. This proves that M/L is a sum of simple modules, and so the result follows from the preceding lemma. □

To show that an algebra is semisimple, we do not want to have to check the condition for every possible module. Fortunately we have

PROPOSITION 2.1.6. *Every finite dimensional A-module is isomorphic to a quotient of A^n for some n. Hence an algebra A is semisimple if and only if A is semisimple as an A-module.*

PROOF. (Sketch) Suppose that M is a finite dimensional A-module, spanned by some elements $m_1, \ldots, m_n$. We define a map

$$\phi : \oplus_{i=1}^{n} A \longrightarrow M$$

by

$$\phi((a_1, \ldots, a_n)) = \sum_{i=1}^{n} a_i m_i.$$

It is easy to check that this is a homomorphism of A-modules, and so by the isomorphism theorem we have that

$$M \cong \oplus_{i=1}^{n} A / \ker \phi.$$

The result now follows from the preceding lemma. □

For finite groups we can say exactly when kG is semisimple:

THEOREM 2.1.7 (Maschke). *Let G be a finite group. Then the group algebra kG is semisimple if and only if the characteristic of k does not divide $|G|$, the order of the group.*

PROOF. (Sketch) First suppose that the characteristic of k does not divide $|G|$. We must show that every kG-submodule M of kG has a complement as a module. Clearly as vector spaces we can find N such that $M \oplus N = kG$. Let $\pi : kG \longrightarrow M$ be the projection map $\pi(m+n) = m$ for all $m \in M$ and $n \in N$. We want to modify π so that it is a module homomorphism, and then show that the kernel is the desired complement.

Define a map $T_\pi : kG \longrightarrow M$ by

$$T_\pi(m) = \frac{1}{|G|} \sum_{g \in G} g(\pi(g^{-1}m)).$$

Note that this is possible as $|G|^{-1}$ exists in k. It is then routine to check that T_π is a kG-module map.

Now let $K = \ker(T_\pi)$, which is a submodule of kG. We want to show that $kG = M \oplus K$. First show that T_π acts as the identity on M, which implies that $M \cap K = 0$. Next note that by the rank-nullity theorem for linear maps, $kG = M + K$. Combining these two facts we deduce that $kG = M \oplus K$ as required.

For the reverse implication, consider $w = \sum_{g \in G} g \in kG$. It is easy to check that every element of g fixes w, and hence w spans a one-dimensional submodule M of kG. Now suppose that there is a complementary submodule N of kG, and decompose $1 = e + f$ where e and f are the idempotents corresponding to M and N respectively. We have $e = \lambda w$ for some $\lambda \in k$, and $e^2 = e = \lambda^2 w^2$. It is easy to check that $w^2 = |G| w$ and hence $\lambda w = \lambda^2 |G| w$ which implies that $1 = \lambda |G|$. But this contradicts the fact that $|G| = 0$ in k. □

The next result will be important in the following section.

LEMMA 2.1.8. *The algebra $M_n(k)$ is semisimple.*

PROOF. Let E_{ij} denote the matrix in $A = M_n(k)$ consisting of zeros everywhere except for the (i, j)th entry, which is 1. We first note that

$$1 = E_{11} + E_{22} + \cdots + E_{nn}$$

is an orthogonal idempotent decomposition of 1, and hence A decomposes as a direct sum of modules of the form AE_{ii}. We will show that these summands are simple.

First observe that AE_{ii} is just the set of matrices which are zero except possibly in column i. Pick $x \in AE_{ii}$ non-zero; we must show that $Ax = AE_{ii}$. As x is non-zero there is some entry x_{mi} in the matrix x which is non-zero. But then

$$E_{jm} x = x_{mi} E_{ji} \in Ax$$

and hence $E_{ji} \in Ax$ for all $1 \leq j \leq n$. But this implies that $Ax = AE_{ii}$ as required. □

2.2. Schur's lemma and the Artin-Wedderburn theorem

We begin with Schur's lemma, which tells us about automorphisms of simple modules.

LEMMA 2.2.1 (Schur). *Let S be a simple A-module and $\phi : S \longrightarrow S$ a non-zero homomorphism. Then ϕ is invertible.*

PROOF. Let $M = \ker \phi$ and $N = \operatorname{im} \phi$; these are both submodules of S. But S is simple and $\phi \neq 0$, so $M = 0$ and ϕ is injective. Similarly we see that $N = S$, so ϕ is surjective, and hence ϕ is invertible. □

LEMMA 2.2.2. *If k is algebraically closed and S is a finite dimensional simple module with non-zero endomorphism ϕ, then $\phi = \lambda . \mathrm{id}_S$, for some non-zero $\lambda \in k$.*

PROOF. As k is algebraically closed and $\dim S < \infty$ the map ϕ has an eigenvalue $\lambda \in k$. Then $\phi - \lambda \, \mathrm{id}_S$ is an endomorphism of S with non-zero kernel (containing all eigenvectors with eigenvalue λ). Arguing as in the preceding lemma we deduce that $\ker(\phi - \lambda \, \mathrm{id}_S) = S$, and hence $\phi = \lambda \, \mathrm{id}_S$. □

Given an A-module M we set

$$\mathrm{End}_A(M) = \{\phi : M \longrightarrow M \mid \phi \text{ is an } A\text{-homomorphism}\}.$$

This is a subalgebra of $\mathrm{End}_k(M)$. More generally, if M and N are A-modules we set

$$\mathrm{Hom}_A(M,N) = \{\phi : M \longrightarrow N \mid \phi \text{ is an } A\text{-homomorphism}\}.$$

Arguing as in the proof of Lemma 2.2.1 above we obtain

LEMMA 2.2.3 (Schur). *If k is algebraically closed and S and T are finite dimensional simple A-modules then*

$$\mathrm{Hom}_A(S,T) \cong \begin{cases} k & \text{if } S \cong T \\ 0 & \text{otherwise.} \end{cases}$$

We can now give a complete classification of the finite dimensional semisimple algebras.

THEOREM 2.2.4 (Artin-Wedderburn). *Let A be a finite dimensional algebra over an algebraically closed field k. Then A is semisimple if and only if*

$$A \cong M_{n_1}(k) \oplus M_{n_2}(k) \oplus \cdots \oplus M_{n_t}(k)$$

for some $t \in \mathbb{N}$ and $n_1, \ldots, n_t \in \mathbb{N}$.

PROOF. (Sketch) We saw in Lemma 2.1.8 that $M_n(k)$ is a semisimple algebra, and if A and B are semisimple algebras, then it is easy to verify that $A \oplus B$ is semisimple.

For the reverse implication suppose that M and N are A-modules, with $M = \oplus_{i=1}^n M_i$ and $N = \oplus_{i=1}^m N_i$. The first claim is that $\mathrm{Hom}_A(M,N)$ can be identified with the space of matrices

$$\{(\phi_{ij})_{1\leq i\leq n, 1\leq j\leq m} \mid \phi_{i,j} : M_j \longrightarrow N_i \text{ an } A\text{-homomorphism}\}$$

and that if $M = N$ with $M_i = N_i$ for all i then this space of matrices is an algebra by matrix multiplication, isomorphic to $\mathrm{End}_A(M)$. This follows by an elementary calculation.

Now apply this to the special case where $M = N = A$, and

$$A = (S_1 \oplus S_2 \oplus \cdots \oplus S_{n_1}) \oplus (S_{n_1+1} \oplus \cdots \oplus S_{n_1+n_2}) \oplus \cdots \oplus (S_{n_1+n_2+\cdots+n_{t-1}+1} \oplus \cdots \oplus S_{n_1+n_2+\cdots+n_t})$$

is a decomposition of A into simples such that two simples are isomorphic if and only if they occur in the same bracketed term. By Schur's Lemma above we see that ϕ_{ij} in this special case is 0 if S_i and S_j are in different bracketed terms, and is some $\lambda_{ij} \in k$ otherwise. There is then an obvious isomorphism of $\mathrm{Hom}_A(A,A)$ with $M_{n_1}(k) \oplus \cdots \oplus M_{n_t}(k)$. Finally, we note that for any algebra A we have

$$\mathrm{End}_A(A,A) \cong A^{op}$$

and hence

$$A = (A^{op})^{op} \cong M_{n_1}(k)^{op} \oplus \cdots \oplus M_{n_t}(k)^{op}.$$

But it is easy to see that $M_n(k) \cong M_n(k)^{op}$ via the map taking a matrix X to its transpose, and so we are done. □

We can also describe all the simple modules for such an algebra.

COROLLARY 2.2.5. *Suppose that*

$$A \cong M_{n_1}(k) \oplus M_{n_2}(k) \oplus \cdots \oplus M_{n_t}(k).$$

Then A has exactly t isomorphism classes of simple modules, one for each matrix algebra. If S_i is the simple corresponding to $M_{n_i}(k)$ then $\dim S_i = n_i$ and S_i occurs precisely n_i times in a decomposition of A into simple modules.

PROOF. (Sketch) Choose a basis for A such that for each element $a \in A$ the map $x \longmapsto ax$ is given by a block matrix

$$\begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & 0 & A_t \end{pmatrix}$$

where $A_i \in M_{n_i}(k)$. Then A is the direct sum of the spaces given by the columns of this matrix, each of dimension n_i. Arguing as in Lemma 2.1.8 we see that each of these column spaces is a simple A-module. Swapping rows in a given block gives isomorphic modules. Thus there are at most t non-isomorphic simples in a decomposition of A (and hence by Proposition 2.1.6 at most t isomorphism classes). Two simples from different blocks cannot be isomorphic (by considering the action of the matrix which is the identity in block A_i and zero elsewhere). $\square$

REMARK 2.2.6. *If k is not algebraically closed then the proofs of Lemmas 2.2.2 and 2.2.3 no longer hold. Instead one deduces that for a simple module S the space* $\mathrm{End}_A(S,S)$ *is a division ring over k. (A* division ring *is a non-commutative version of a field.) There is then a version of the Artin-Wedderburn theorem, but where each $M_n(k)$ is replaced by some $M_n(D_i)$ with D_i some division ring containing k.*

2.3. The Jacobson radical

Suppose that A is not a semisimple algebra. One way to measure how far from semisimple it is would be to find an ideal I in A such that A/I is semisimple and I is minimal with this property.

DEFINITION 2.3.1. *The* Jacobson radical *(or just* radical*) of an algebra A, denoted $\mathscr{J}(A)$ (or just $\mathscr{J}$), is the set of elements $a \in A$ such that $aS = 0$ for all simple modules S. It is easy to verify that this is an ideal in A.*

DEFINITION 2.3.2. *An ideal is* nilpotent *if there exists n such that $I^n = 0$. A* maximal submodule *in a module M is a module $L \subset M$ which is maximal by inclusion. The* annihilator $\mathrm{Ann}(M)$ *of a module M is the set of $a \in A$ such that $aM = 0$. This is easily seen to be a submodule of A.*

When discussing the Jacobson radical, the following result is useful.

LEMMA 2.3.3. *Let A be a finite dimensional algebra. Then A has a largest nilpotent ideal.*

PROOF. Consider the set of nilpotent ideals in A, and chose one, I, of maximal dimension. If J is another nilpotent ideal then the ideal $I+J$ is also nilpotent. (If $I^n = 0$ and $J^m = 0$ then $(I+J)^{m+n} = 0$, as the expansion of any expression $(a+b)^{n+m}$ with $a \in I$ and $b \in J$ contains at least n copies of a or m copies of b.) But then $\dim(I+J) = \dim I$ and hence $J \subseteq I$. $\square$

THEOREM 2.3.4 (Jacobson). *Let A be a finite dimensional algebra. The ideal $\mathscr{J}(A)$ is*
(a) the largest nilpotent ideal N in A.
(b) the intersection D of all maximal submodules of A.
(c) the smallest submodule R of A such that A/R is semisimple.

PROOF. (a) First suppose that S is simple. Then NS is a submodule of S. If $NS = S$ then by induction $N^m S = S$ for all $m \geq 1$. But this contradicts the nilpotency of N, and so $N \subseteq \mathscr{J}$. For the reverse inclusion, consider a composition series for A

$$0 = A_n \subset A_{n-1} \subset \cdots \subset A_0 = A.$$

As A_i/A_{i+1} is simple we have $a(A_i/A_{i+1}) = 0$ for all $a \in \mathscr{J}$. But this implies that $\mathscr{J} A_i \subseteq A_{i+1}$, and hence

$$\mathscr{J}^n \subset \mathscr{J}^n A \subset A_n = 0.$$

(b) Suppose that $a \in \mathscr{J}$ and M is a maximal submodule of A. Then A/M is simple and so $a(A/M) = 0$. In other words, $a(1+M) = 0+M$ and so $a \in M$. Thus $\mathscr{J} \subset M$ for every maximal submodule of A.

For the reverse inclusion, suppose that $\mathscr{J} \not\subseteq D$. Then there exists some simple S and $s \in S$ with $Ds \neq 0$. Now Ds is a submodule of S, and hence $Ds = S$. Thus there exists $d \in D$ with $ds = s$; so $d - 1 \in \mathrm{Ann}(S) \not\subseteq A$, and there exists a maximal submodule M of A with $\mathrm{Ann}(S) \subseteq M$. But then $d \in D \subseteq M$ and $1 - d \in M$ implies that $1 \in M$, which contradicts $M \subset A$.

(c) (Sketch) First we claim that D can be expressed as the intersection of finitely many maximal submodules of A. To see this pick some submodule L which is the intersection of finitely many maximal submodules, such that $\dim L$ is minimal. Clearly $D \subseteq L$. For any maximal M in A we must have that $L = L \cap M$, and hence $L \subseteq D$.

Thus $D = M_1 \cap M_2 \cap \ldots \cap M_n$ for some maximal submodules $M_1, \ldots M_n$. There is a homomorphism

$$\phi : A/D \longrightarrow A/M_1 \oplus \cdots A/M_n$$

given by $\phi(a) = (a + M_1, \ldots, a + M_n)$. It is easy to see this is injective. As each M_i is maximal we have embedded A/D into a semisimple module, and hence A/D is semisimple by Lemma 2.1.5.

Now suppose that A/X is semisimple. It remains to show that $D \subseteq X$. Write A/X as a direct sum of simples $S_i = L_i/X$. Then it is easy to check that the submodule $M_i = \sum_{i \neq j} L_i$ is a maximal submodule of A, and that the intersection of the M_i equals X. By definition this intersection contains D, as required. □

The Jacobson radical can be used to understand the structure of A-modules:

LEMMA 2.3.5 (Nakayama). *If M is a finite dimensional A-module such that $\mathscr{J} M = M$ then $M = 0$.*

PROOF. (Sketch, for the case A is finite dimensional) Suppose that $M \neq 0$ and choose a minimal set of generators $m_1, \ldots, m_t$ of M as an A-module. Now $m_t \in M = \mathscr{J} M$ implies that

$$m_t = \sum_{i=1}^{t} a_i m_i$$

for some $a_i \in \mathscr{J}$, and so

$$(1 - a_t)m_t = \sum_{i=1}^{t-1} a_i m_i.$$

Now $a_t \in \mathscr{J}$ implies that a_t is nilpotent, and then it is easy to check that $1 - a_t$ must be invertible. But this implies that m_t can be expressed in terms of the remaining m_i, which contradicts minimality. $\square$

We have the following generalisation of Nakayama's Lemma.

PROPOSITION 2.3.6. *If A is a finite dimensional algebra and M is a finite dimensional A-module then $\mathscr{J}M$ equals*
(a) the intersection D of all maximal submodules of M.
(b) the smallest submodule R of M such that M/R is semisimple.

PROOF. (Sketch) Suppose that M_i is a maximal submodule of M. Then M/M_i is simple, and hence by Nakayama's lemma $\mathscr{J}(M/M_i) = 0$. Therefore $\mathscr{J}M \subseteq \mathscr{J}M_i \subseteq M_i$ and so $\mathscr{J}M \subseteq D$.

By Theorem 2.3.4 the module $M/\mathscr{J}M$ is semisimple, as it is a module for $A/\mathscr{J}$. Now suppose that L is a submodule of M such that M/L is semisimple. Let $M/L = M_1/L \oplus \cdots \oplus M_t/L$ where each M_i/L is simple. Then the modules $N_j = \sum_{i \neq j} M_i$ are maximal submodules of M and L is the intersection of the N_j. Hence $\mathscr{J}M$ is a submodule of L as $\mathscr{J}M$ is a submodule of D. Taking $L = \mathscr{J}M$ we see that D is a submodule of $\mathscr{J}M$ which completes the proof. $\square$

Motivated by the last result, we have

DEFINITION 2.3.7. *The* radical *of a module M is defined to be the module $\mathscr{J}M$. Note that when $M = A$ this agrees with the earlier definition of the radical of an algebra. The* head *or* top *of M, denoted* $\mathrm{hd}(M)$ *or* $\mathrm{top}(M)$, *is the module $M/\mathscr{J}M$. By the last proposition the sequence*

$$M \supset \mathscr{J}M \supset \mathscr{J}^2M \supset \cdots \supset \mathscr{J}^tM \supset \mathscr{J}^{t+1}M = 0$$

is such that each successive quotient is the largest semisimple quotient possible. This is called the Loewy series *for M, and $t+1$ is the* Loewy length *of M.*

The head of a module M is the largest semisimple quotient of M. It can be shown that the submodule of M generated by all simple submodules is the largest semisimple submodule of M; we call this the *socle* of M, and denote it by $\mathrm{soc}(M)$.

2.4. The Krull-Schmidt theorem

Given a finite dimensional A-module M, it is clear that we can decompose M as a direct sum of indecomposable modules. The Krull-Schmidt theorem says that this decomposition is essentially unique, and so it is enough to classify the indecomposable modules for an algebra.

THEOREM 2.4.1 (Krull-Schmidt). *Let A be a finite dimensional algebra and M be a finite dimensional A-module. If*

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_n = N_1 \oplus N_2 \oplus \cdots \oplus N_m$$

are two decompositions of M into indecomposables then $n = m$ and there exists a permutation σ of $\{1, \ldots n\}$ such that $N_i \cong M_{\sigma(i)}$.

PROOF. The idea is to proceed by induction on n, at each stage cancelling out summands which are known to be isomorphic. The details are slightly technical, and so will be omitted here. Instead we will review below some of the ideas used in the proof. $\square$

A key idea in the proof of the Krull-Schmidt theorem is the notion of a local algebra.

DEFINITION 2.4.2. *An algebra A is* local *if it has a unique maximal right (or left) ideal.*

There are various characterisations of a local algebra.

LEMMA 2.4.3. *Suppose that A is a finite dimensional algebra over an algebraically closed field. Then the following are equivalent:*
(a) A is a local algebra.
(b) The set of non-invertible elements of A form an ideal.
(c) The only idempotents in A are 0 and 1.
(d) The quotient $A/\mathscr{J}$ is isomorphic to k.

PROOF. This is not difficult, but is omitted as it requires a few preparatory results. $\square$

REMARK 2.4.4. *In fact (a) and (b) are equivalent for any algebra A. However, there exist examples of infinite dimensional algebras with only 0 and 1 as idempotents which are not local, for example $k[x]$. Also, if the field is not algebraically closed then $A/\mathscr{J}$ will only be a division ring in general.*

LEMMA 2.4.5 (Fitting). *Let M be a finite dimensional A-module, and $\phi \in \mathrm{End}_A(M)$. Then for large enough n we have*

$$M = \mathrm{im}(\phi^n) \oplus \ker(\phi^n).$$

In particular, if M is indecomposable then any non-invertible endomorphism of M must be nilpotent.

PROOF. Note that $\phi^{i+1}(M) \subseteq \phi^i(M)$ for all i. As M is finite dimensional there must exist an n such that $\phi^{n+t}(M) = \phi^n(M)$, for all $t \geq 1$ and so ϕ^n is an isomorphism from $\phi^n(M)$ to $\phi^{2n}(M)$. For $m \in M$ let x be an element such that $\phi^n(m) = \phi^{2n}(x)$. Now

$$m = \phi^n(x) + (m - \phi^n(x)) \in \mathrm{im}(\phi^n) + \ker(\phi^n)$$

and so $M = \mathrm{im}(\phi^n) + \ker(\phi^n)$. If $\phi^n(m) \in \mathrm{im}(\phi^n) \cap \ker(\phi^n)$ then $\phi^{2n}(m) = 0$, and so $\phi^n(m) = 0$. Thus the sum is direct, as required. $\square$

Local algebras are useful as they allow us to detect indecomposable modules.

LEMMA 2.4.6. *Let M be a finite dimensional A-module. Then M is indecomposable if and only if $\mathrm{End}_A(M)$ is a local algebra.*

PROOF. First suppose that $M = M_1 \oplus M_2$, and for $i = 1,2$ let e_i be the map from M to M which maps $m_1 + m_2$ to m_i. Then $e_i \in \mathrm{End}_A(M)$ is non-invertible (as it has non-zero kernel). But $e_1 + e_2 = 1$, which is invertible, which implies that $\mathrm{End}_A(M)$ is not local by Lemma 2.4.3.

Now suppose that M is indecomposable. Let I be a maximal right ideal in $\mathrm{End}_A(M)$, and pick $\phi \in \mathrm{End}_A(M) \backslash I$. By maximality we have $\mathrm{End}_A(M) = \mathrm{End}_A(M)\phi + I$. Thus we can write $1 = \theta\phi + \mu$ where $\theta \in \mathrm{End}_A(M)$ and $\mu \in I$. Note that any element in I cannot be an isomorphism of M (as it would then be invertible), and hence by Fitting's Lemma we have that $\mu^n = 0$ for some $n >> 0$. But then

$$(1+\mu+\mu^2+\ldots+\mu^{n-1})\theta\phi = (1+\mu+\mu^2+\ldots+\mu^{n-1})(1-\mu) = 1-\mu^n = 1$$

and so ϕ is an isomorphism. But then I consists precisely of the non invertible elements in $\mathrm{End}_A(M)$, and the result follows by Lemma 2.4.3. □

2.5. Exercises

(1) Let $A = k[x]$ and M be the 2-dimensional A module where x acts via the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

with respect to some basis of M. Prove that M is not a semisimple module.

(2) Prove the assertion in the proof of the Artin-Wedderburn theorem that

$$\mathrm{End}_A(A,A) \cong A^{op}.$$

(3) The *centre* of an algebra A, denoted $Z(A)$, is the set of $z \in A$ such that $za = az$ for all $a \in A$. This is a subalgebra of A. If k is algebraically closed and S is a simple A-module show that for all $z \in Z(A)$ there exists $\lambda \in k$ such that $zm = \lambda m$ for all $m \in S$.

(4) Show that $k[x]/(x^n)$ is a local algebra.

(5) Let G be a finite group of order p^n, and k be a field of characteristic p.
   (a) Prove that the ideal I generated by the set

$$\{1-g : g \in Z(G)\}$$

   is nilpotent in kG.
   (b) Show that I is the kernel of some map from kG to $k(G/Z(G))$.
   (c) Deduce that kG is local. You may use the fact that for all such G we have $Z(G) \neq 1$.

(6) Show that $k[x,y]$ is not a local algebra, but only contains the two idempotents 0 and 1. This demonstrates the need for finite dimensionality in Lemma 2.4.3.