CHAPTER 3

Projective and injective modules

3.1. Projective and injective modules

DEFINITION 3.1.1. *A* short exact sequence *of A-modules is a sequence of the form*

$$0 \longrightarrow L \xrightarrow{\phi} M \xrightarrow{\psi} N \to 0$$

such that the map ϕ is injective, the map ψ is surjective, and $\operatorname{im}\phi = \ker\psi$*. More generally, a sequence*

$$\cdots \longrightarrow L \xrightarrow{\phi} M \xrightarrow{\psi} N \longrightarrow \cdots$$

is exact at M *if* $\operatorname{im}\phi = \ker\psi$*. If a sequence is exact at every module then it is called* exact*. (Thus a short exact sequence is exact.)*

Note that in a short exact sequence as above we have that

$$M/L \cong N$$

by the isomorphism theorem, and $\dim M = \dim L + \dim N$. When a sequence starts or ends in a 0 it is common to assume that it is exact (as we will do in what follows).

LEMMA 3.1.2. *Given a short exact sequence of A-modules*

$$0 \longrightarrow L \xrightarrow{\phi} M \xrightarrow{\psi} N \to 0$$

the following are equivalent:
(a) There exists a homomorphism $\theta : N \longrightarrow M$ such that $\psi\theta = id_N$.
(b) There exists a homomorphism $\tau : M \longrightarrow L$ such that $\tau\phi = id_L$.
(c) There is a module X with $M = X \oplus \ker(\psi)$.

PROOF. (Sketch) We will show that (a) is equivalent to (c); that (b) is equivalent to (c) is similar. First suppose that θ as in (a) exists. Then θ must be an injective map. Let $X = \operatorname{im}(\theta)$, a submodule of M isomorphic to N. It is easy to check that $X \cap \ker(\psi) = 0$ and that $\dim(X \oplus \ker(\psi)) = \dim M$ by exactness at M. Therefore $M = X \oplus \ker(\psi)$.

Now suppose that $M = X \oplus \ker(\psi)$. Consider the restriction of ψ to X; it is clearly an isomorphism and so θ can be taken to be an inverse to ψ. □

DEFINITION 3.1.3. *An A-module P is* projective *if for all surjective A-module homomorphisms $\theta : M \longrightarrow N$ and for all $\phi : P \longrightarrow N$ there exists $\psi : P \longrightarrow M$ such that $\theta\psi = \phi$.*

Thus a module P is projective if there always exists ψ such that the following diagram commutes

$$\begin{tikzcd} & P \arrow[dl, dotted, "\psi"'] \arrow[d, "\phi"] & \\ M \arrow[r, "\theta"] & N \arrow[r] & 0. \end{tikzcd}$$

(Note that here we are using our convention about exactness for the bottom row in the diagram.)

There is a dual definition, obtained by reversing all the arrows and swapping surjective and injective.

DEFINITION 3.1.4. *An A-module I is* injective *if for all injective A-module homomorphisms $\theta : N \longrightarrow M$ and for all $\phi : N \longrightarrow I$ there exists $\psi : M \longrightarrow I$ such that $\psi\theta = \phi$.*

Thus a module I is injective if there always exists ψ such that the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\theta"] \arrow[d, "\phi"] & N \arrow[dl, dotted, "\psi"] \\ & I & \end{tikzcd}$$

EXAMPLE 3.1.5. *For $m \geq 1$ the module A^m is projective. To see this, denote the ith coordinate vector $(0,\ldots,0,1,0\ldots,0)$ by v_i, and suppose that $\phi(v_i) = n_i \in N$. As g is surjective, there exists $m_i \in M$ such that $g(m_i) = n_i$. Given a general element $(a_1,\ldots,a_m) \in A^m$ define*

$$\psi(a_1,\ldots,a_m) = \sum_{i=1}^{m} a_i m_i.$$

It is easy to verify that this gives the desired A-module homomorphism.

We would like a means to recognise projective modules P without having to consider all possible surjections and morphisms from P. The following lemma provides this, and shows that the above example is typical.

LEMMA 3.1.6. *For an algebra A the following are equivalent.*
(a) P is projective.
(b) Whenever $\theta : M \longrightarrow P$ is a surjection then $M \cong P \oplus \ker(\theta)$.
(c) P is isomorphic to a direct summand of A^m for some m.

PROOF. First suppose that P is projective. We have by definition a commutative diagram

$$\begin{tikzcd} & P \arrow[dl, dotted, "\psi"'] \arrow[d, "\mathrm{id}_P"] & \\ M \arrow[r, "\theta"] & P \arrow[r] & 0. \end{tikzcd}$$

and by Lemma 3.1.2 this implies that $M \cong P \oplus \ker(\theta)$.

Now suppose that (b) holds. Given any A-module M with generators m_i, $i \in I$, there is a surjection from $\oplus_{i\in I} A$ onto M given by the map taking 1 in the ith copy of A to m_i. Taking $M = P$ a projective we deduce that (c) holds.

Finally suppose that $A^m \cong P \oplus X$ for some P and X. Let π be the projection map from A^n onto P, and ι be the inclusion map from P into A^m. Given modules M and N and a surjection from M to N we have the commutative diagram

$$\begin{tikzcd}
 & A^n \arrow[d, shift left, "\pi"] \arrow[ddl, dotted, "\psi_1"'] & \\
 & P \arrow[u, shift left, "\iota"] \arrow[d, "\phi"] \arrow[dl, dotted, "\psi_1\iota"'] & \\
M \arrow[r, "\theta"] & P \arrow[r] & 0.
\end{tikzcd}$$

It is easy to check that $\psi_1\iota$ gives the desired map ψ for P in the definition of a projective module. $\square$

Suppose that A is finite dimensional and

$$A = P(1) \oplus \cdots \oplus P(n) \tag{1}$$

is a decomposition of A into indecomposable direct summands. By the last result these summands are indecomposable projective modules.

LEMMA 3.1.7. *Suppose that A is a finite dimensional algebra. Let P be a projective A-module with submodule N, and suppose that every homomorphism $\phi : P \longrightarrow P$ maps N to N. Then there is a surjection from* $\mathrm{End}_A(P)$ *onto* $\mathrm{End}_A(P/N)$ *and if P is indecomposable then so is P/N.*

PROOF. (Sketch) Given $\phi : P \longrightarrow P$ let $\bar{\phi}$ be the obvious map from P/N to P/N. Check this is well-defined; it is clearly a homomorphism. The map $\phi \longrightarrow \bar{\phi}$ gives an algebra homomorphism from $\mathrm{End}_A(P)$ to $\mathrm{End}_A(P/N)$; given $\psi \in \mathrm{End}_A(P/N)$ use the projective property of P to construct a map ϕ so that $\bar{\phi} = \psi$.

If P is indecomposable then $\mathrm{End}_A(P)$ is a local algebra by Lemma 2.4.6. Therefore there is a unique maximal right ideal in $\mathrm{End}_A(P)$, and hence a unique maximal ideal in $\mathrm{End}_A(P/N)$ (as we have shown that this is a quotient of $\mathrm{End}_A(P)$). Thus $\mathrm{End}_A(P/N)$ is local, and hence P/N is indecomposable. $\square$

THEOREM 3.1.8. *Let A be a finite dimensional algebra, and decompose A as in (1). Setting $S(i) = P(i)/\mathscr{J}P(i)$ we have*
(a) The module $S(i)$ is simple, and every simple A-module is isomorphic to some $S(i)$.
(b) We have $S(i) \cong S(j)$ if and only if $P(i) \cong P(j)$.

PROOF. (Sketch) (a) The modules $S(i)$ is semisimple, so it is enough to check it is indecomposable. Note that $P(i)$ and $\mathscr{J}P(i)$ satisfy the assumptions in Lemma 3.1.7, and so $S(i)$ is indecomposable.

Let S be a simple module and choose $x \neq 0$ in S. As $1x = x$ there is some $P(i)$ such that $P(i)x \neq 0$ (as $Ax \neq 0$ and A is the direct sum of the $P(i)$). Define a homomorphism from $P(i)$ to S, so S is a simple quotient of $P(i)$. But $\mathscr{J}P(i)$ is the unique maximal submodule of $P(i)$, and so as S is simple we have $S \cong P(i)/\mathscr{J}P(i)$.

(b) If $P(i) \cong P(j)$ via ϕ it is easy to see that $\phi(\mathscr{J}P(i)) \subseteq \mathscr{J}P(j))$. Hence ϕ induces a homomorphism from $S(i)$ to S_j. As ϕ is invertible this has an inverse, and so $S(i) \cong S(j)$ by Schur's Lemma.

If $S(i) \cong S(j)$ then use the projective property to construct a homomorphism ψ from $P(i)$ to $P(j)$. Show that the image of this map cannot be inside $\mathscr{J}P(j)$, so as $\mathscr{J}P(j)$ is a maximal submodule ψ must have image all of $P(j)$. By Lemma 3.1.6 we deduce that $P(i) \cong P(j) \oplus \ker(\psi)$, and so as $P(i)$ is indecomposable we have $\ker(\psi) = 0$. Thus ψ is an isomorphism. □

By Krull-Schmidt, this implies that a finite dimensional algebra A has only finitely many isomorphism classes of simple modules.

DEFINITION 3.1.9. *Let M be a finite dimensional A-module. A* projective cover *for M is a projective module P such that*

$$P/\mathscr{J}P \cong M/\mathscr{J}M$$

and there exists a surjection $\pi : P \longrightarrow M$.

LEMMA 3.1.10. *Let A be a finite dimensional algebra. Every finite dimensional A-module has a projective cover, which is unique up to isomorphism. In particular, suppose that*

$$M/\mathscr{J}M \cong S(1)^{n_1} \oplus S(2)^{n_2} \oplus \cdots \oplus S(t)^{n_t}.$$

Then

$$P = P(1)^{n_1} \oplus P(2)^{n_2} \oplus \cdots \oplus P(t)^{n_t}$$

is a projective cover of M via the canonical surjection on each component.

PROOF. (Sketch) It is clear that the given P satisfies $P/\mathscr{J}P \cong M/\mathscr{J}M$. Use the projective property to construct a homomorphism π from P to M; it is easy to see that $\mathrm{im}(\pi) + \mathscr{J}M = M$ by the commutativity of the related diagram. But then $\mathscr{J}(M/\mathrm{im}(\pi)) = (\mathscr{J}M + \mathrm{im}(\pi))/\mathrm{im}(\pi) = M/\mathrm{im}(\pi)$ and so by Nakayama's Lemma we have $M/N = 0$. Thus π is surjective as required. □

DEFINITION 3.1.11. *A* projective resolution *of a module M is an exact sequence*

$$\cdots \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow M \longrightarrow 0$$

such that all the P_i are projective.

By induction using Lemma 3.1.10 we deduce

PROPOSITION 3.1.12. *If A is a finite dimensional algebra then every finite dimensional A-module has a projective resolution.*

There is a similar theory for injective modules, but instead of developing this separately we will instead use dual modules to relate the two.

The injective analogue of a projective cover is called the *injective envelope* of M. An *injective resolution* of M is an exact sequence

$$0 \longrightarrow M \longrightarrow I_1 \longrightarrow I_2 \longrightarrow I_3 \longrightarrow \cdots$$

such that all the I_i are injective.

THEOREM 3.1.13. *Suppose that A is a finite dimensional algebra and M a finite dimensional A-module.*
(a) M is simple if and only if M^* is a simple A^{op}-module.
(b) M is projective if and only if M^* is an injective A^{op}-module.
(c) M is injective if and only if M^* is a projective A^{op}-module.
(d) The injective envelope of M is I if and only if the projective cover of the A^{op}-module M^* is I^*.

PROOF. This is a straightforward application of duality. □

Projective and injective modules play a crucial role in the study of the cohomology of representations. In a non-semisimple representation theory there are certain spaces associated to $\mathrm{Hom}_A(M,N)$ called *extension groups* $\mathrm{Ext}^i_A(M,N)$. To introduce these properly, we would need to work with the category of modules, and introduce the notion of a derived functor. Unfortunately this is beyond the scope of the current course.

3.2. Idempotents and direct sum decompositions

Every algebra has at least two idempotents, 0 and 1. If A is not local then there exists another idempotent $e \in A$ and e and $1-e$ are two non-zero orthogonal idempotents, giving a decomposition of A-modules

$$A = Ae \oplus A(1-e).$$

If e is a central idempotent then so is $1-e$, and the above decomposition becomes a direct sum of algebras. Conversely, if $A = M_1 \oplus M_2$ as an A-module, then the corresponding decomposition $1 = e_1 + e_2$ is an orthogonal idempotent decomposition of 1. If the decomposition of A is as a direct sum of algebras, then the corresponding idempotents are central.

DEFINITION 3.2.1. *We say that an algebra is* connected *or* indecomposable *if* 0 *and* 1 *are the only central idempotents in A.*

Note that if A is not connected, say $A = A_1 \oplus A_2$, then any A-module M decomposes as a direct sum $M_1 \oplus M_2$ where M_i is an A_i-module for $i = 1,2$. (This follows by decomposing $1 \in A$ and applying it to M.) Thus we can reduce the study of the representations of an algebra to the case where the algebra is connected.

Suppose that A is a finite dimensional algebra. By repeatedly decomposing A as an A-module we can write

$$A = P_1 \oplus \cdots \oplus P_n$$

where the P_i are indecomposable left ideals in A. (The sum is finite as A is finite dimensional.) There is a corresponding decomposition of 1 as a sum of primitive orthogonal idempotents. Conversely any such decomposition of 1 gives rise to a decomposition of A into indecomposable left ideals. Note that we can identify primitive idempotents by the following application of Lemma 2.4.6

COROLLARY 3.2.2. *Suppose that A is a finite dimensional algebra. Then an idempotent $e \in A$ is primitive if and only if eAe is local.*

DEFINITION 3.2.3. *Suppose that A is a finite dimensional algebra with a complete set $\{e_1, \ldots e_n\}$ of primitive orthogonal idempotents. Then A is* basic *if $Ae_i \cong Ae_j$ implies that $i = j$.*

Basic algebras have the following nice properties.

PROPOSITION 3.2.4. *Suppose that k is algebraically closed.*
(a) A finite dimensional k-algebra A is basic if and only if

$$A/\mathscr{J} \cong k \times k \times \cdots \times k.$$

(b) Every simple module over a basic algebra is one dimensional.

PROOF. (Sketch) (a) Suppose that A is basic and consider a complete set of primitive idempotents $e_1, \ldots, e_n$ for A. By Theorem 3.1.8 the modules $S_i = (A/\mathscr{J})e_i$ are simple $A/\mathscr{J}$-modules. Also, as A is basic these simples are non-isomorphic. Then Schur's lemma implies that Hom-spaces between such simples are isomorphic to 0 or k, and one can define an injective homomorphism

$$\phi : A/\mathscr{J} \longrightarrow \mathrm{End}_{A/\mathscr{J}}(S_1 \oplus \cdots \oplus S_n) \cong k \times \cdots \times k.$$

By dimensions this is an isomorphism.

If $A/\mathscr{J}$ is basic then the S_i above are all non-isomorphic (as the primitive idempotents are even central in $A/\mathscr{J}$), and the same argument as above implies that A is basic.

(b) Any simple A-module is also an $A/\mathscr{J}$-module by Nakayama's Lemma. But by part (a) this is isomorphic to $k \times \cdots \times k$, which implies the result. □

Given an arbitrary finite dimensional algebra A, we can associate a basic algebra to it in the following manner.

DEFINITION 3.2.5. *Suppose that A is finite dimensional and has a complete set $\{e_1, \ldots e_n\}$ of primitive orthogonal idempotents. Pick idempotents $e_{i_1}, \ldots e_{i_t}$ from this set such that $Ae_{i_a} \cong Ae_{i_b}$ implies that $a = b$, and so that the collection is maximal with this property. Then define*

$$e_A = \sum_{a=1}^{t} e_{i_a}$$

and set $A^b = e_A A e_A$, the basic algebra associated to *A. (It is easy to see that this is indeed a basic algebra, and is independent of the choice of idempotents.)*

As we have not given a precise definition of a category, we shall state the next result without proof.

THEOREM 3.2.6. *Suppose that A is a finite dimensional algebra. Then the category of finite dimensional A-modules is equivalent to the category of finite dimensional A^b-modules.*

This means that to understand the representation theory of a finite dimensional algebra it is enough to consider representations of the corresponding basic algebra.

Let us now consider the special case of the path algebra of a quiver.

LEMMA 3.2.7. *Let Q be a finite quiver. Then the sum*

$$1 = \sum_{i \in Q_0} \varepsilon_i$$

is a decomposition into a complete set of primitive orthogonal idempotents for kQ.

PROOF. All that remains to prove is that the ε_i are primitive, for which it is enough to show that $\varepsilon_i kQ \varepsilon_i$ is local. Suppose that $e \in \varepsilon_i kQ \varepsilon_i$ is an idempotent. Then $e = \lambda \varepsilon_i + w$ where $\lambda \in k$ and w is a sum of paths from a to a. But then

$$0 = e^2 - e = (\lambda^2 - \lambda)\varepsilon_i + w^2 + (2\lambda - 1)w$$

implies that $w = 0$ and $\lambda = 0$ or $\lambda = 1$. □

We can now characterise the connected path algebras of quivers.

LEMMA 3.2.8. *Let Q be a finite quiver. Then kQ is connected if and only if Q is a connected quiver.*

PROOF. (Sketch) It can be shown that kQ is connected if and only if there does not exist a partition $Q_0 = X \cup Y$ of the set of vertices such that for all $x \in X$ and $y \in Y$ we have $\varepsilon_x kQ \varepsilon_y = 0 = \varepsilon_y kQ \varepsilon_x$.

Clearly if Q is not connected then there exists a partition $Q_0 = X \cup Y$ so that there is no path from a vertex in X to a vertex in Y (or vice versa). Thus in this case $\varepsilon_x kQ \varepsilon_y = 0 = \varepsilon_y kQ \varepsilon_x$, and kQ is not connected.

If kQ is not connected but Q is connected, there exists a partition $Q_0 = X \cup Y$ as above, and elements $x \in X$ and $y \in Y$ with an arrow $\alpha : x \longrightarrow y$ in Q. But then $\alpha \in \varepsilon_x kQ \varepsilon_y$ which contradicts our assumption on kQ. $\square$

THEOREM 3.2.9. *Let Q be a finite, connected, acyclic quiver. Then kQ is a basic connected algebra with radical given by the arrow ideal.*

PROOF. (Sketch) By Lemma 3.2.7 we have a decomposition

$$kQ/R_Q = \bigoplus_{a,b \in Q_o} \varepsilon_a (kQ/R_Q) \varepsilon_b.$$

As R contains all non-trivial paths each summand is non-zero only when $a = b$, in which case it is isomorphic to k. Thus we will be done by Proposition 3.2.4 and Lemma 3.2.8 if we can show that $R_Q = \mathscr{J}$. But as Q is acyclic there exists a maximal path length in Q. Hence $R_Q^n = 0$ for $n >> 0$, and so $R \subseteq \mathscr{J}$ by Theorem 2.3.4. It is not too hard to show that in fact any nilpotent ideal I such that A/I is a product of copies of k must equal $\mathscr{J}(A)$. $\square$

Now we consider the case of bound quiver algebras.

PROPOSITION 3.2.10. *Let Q be a finite quiver with admissible ideal I in kQ. Then*
(a) The set

$$\{e_i = \varepsilon_i + I : i \in Q_0\}$$

is a complete set of primitive orthogonal idempotents in kQ/I.
(b) The algebra kQ/I is connected if and only if Q is a connected quiver.
(c) The algebra kQ/I is basic, with radical R_Q/I.

PROOF. (Sketch) The proofs of (a) and (b) are similar to those for kQ. Part (c) is almost immediate from the corresponding result for kQ. $\square$

We have seen that the representation theory of finite dimensional algebras reduces to the study of connected basic algebras. The last result says that bound quiver algebras for connected quivers are such algebras. We conclude this section with

THEOREM 3.2.11. *Let A be a basic, connected, finite dimensional k-algebra over an algebraically closed field. Then there is a connected quiver Q associated to A and an admissible ideal I in kQ such that*

$$A \cong kQ/I.$$

Thus over algebraically closed fields the study of finite dimensional algebras can be reduced to the study of bound quiver algebras.

PROOF. (Sketch) Rather than give a detailed proof, we will sketch how to construct the quiver associated to A.

Let $\{e_1,\ldots,e_n\}$ be a complete set of primitive orthogonal idempotents in A. Then Q has vertex set $\{1,\ldots,n\}$. Given $1 \leq i, j \leq n$, the number of arrows from i to j equals the dimension of the vector space $e_i(\mathscr{J}/\mathscr{J}^2)e_j$.

One then checks that this quiver is independent of the choice of idempotents and is connected. Then one defines a homomorphism from kQ to A, and show that this is (i) surjective, and (ii) has kernel which is an admissible ideal in kQ. The result then follows from the first isomorphism theorem. □

For a discussion of what happens when k is not algebraically closed see [**Ben91**, Section 4.1].

3.3. Simple and projective modules for bound quiver algebras

In general it is hard to determine explicitly the simple modules for an algebra. Indeed, some of the most important open questions in representation theory relate to determining simple modules. However, in the case of a bound quiver algebra the simple modules can be written down entirely explicitly.

We will also see that the indecomposable projectives can also be easily constructed. The same is true for indecomposable injectives, but we will not consider these in detail here.

Let kQ/I be a bound quiver algebra. We know by Proposition 3.2.10 and Theorem 3.1.8 that the simple modules are parameterised by the vertices of Q, and are all one dimensional (as the algebras are basic). Given this, the following result is almost clear.

PROPOSITION 3.3.1. *Let kQ/I be a bound quiver algebra. For $a \in Q_0$, let $S(a)$ be the representation of Q such that*

$$S(a)_b = \begin{cases} k & a = b \\ 0 & a \neq b \end{cases}$$

and for all arrows α the map $\phi_\alpha = 0$. Then

$$\{S(a) : a \in Q_0\}$$

is a complete set of non-isomorphic simple modules for kQ/I.

PROOF. The only thing that remains to check is that the various simples are not isomorphic, but this is straightforward. □

The description of the projective modules $P(a)$ is slightly more complicated.

PROPOSITION 3.3.2. *Let kQ/I be a bound quiver algebra, and $P(a)$ the projective corresponding to ε_a. Then $P(a)$ can be realised in the following manner.*
For $b \in Q_0$ let $P(a)_b$ be the k-vector space with basis the set of all elements of the form $w+I$ where w is a path from a to b. Given an arrow $\alpha : b \longrightarrow c$, the map $\phi_\alpha : P(a)_b \longrightarrow P(a)_c$ is given by left multiplication by $\alpha + I$.

PROOF. This is a straightforward consequence of the explicit identification of quiver representations with kQ/I-modules given earlier. $\square$

The description of injective modules for a bound quiver algebra is similar, using Theorem 3.1.13.

3.4. Exercises

(1) Let A be an algebra containing e an idempotent, and let M be a left A-module.

(a) Show that eAe is an algebra, and eM is a left eAe-module.
(b) Show that $\mathrm{Hom}_A(Ae, M)$ is a left eAe-module where the action of $a \in eAe$ on a morphism $\phi \in \mathrm{Hom}_A(Ae, M)$ is given by

$$a\phi(-) = \phi(a-).$$

(c) Show that there is an isomorphism of left eAe-modules $eM \cong \mathrm{Hom}_A(Ae, M)$.

(2) A first course on representation theory often considers only representations of finite groups over $\mathbb{C}$. In this case much can be learnt from the study of *characters*. Given a finite group G, a representation V of dimension n can be described by giving a group homomorphism $\rho : G \longrightarrow \mathrm{End}(V)$. By choosing a basis of V we obtain a map from G into $\mathrm{GL}_n(\mathbb{C})$. We define the *character* of V to be the map $\chi_V : G \longrightarrow \mathbb{C}$ given by $\chi_V(g) = \mathrm{Tr}(\rho(g))$, the trace of the matrix $\rho(g)$. This looks like it throws away a lot of information; however this exercise will show that it is still a powerful tool.

(a) Show that the character of V does not depend on the chosen basis.
(b) Show that if V and W are two isomorphic representations of G then $\chi_V = \chi_W$. Hint: Let ϕ be an isomorphism from V to W. Pick a basis for V and consider the corresponding basis of W obtained via ϕ. Now compare the actions of $g \in G$ on each basis.
(c) Suppose that V and W are two simple non-isomorphic representations of G. By the Artin-Wedderburn Theorem $\mathbb{C}G$ is isomorphic to a direct sum of matrix algebras, and there is a corresponding idempotent decomposition $1 = \sum e_i$. Show that there exists i such that e_i acts as the identity on V and as 0 on W. (You may wish to recall Corollary 2.2.5.)
(d) Deduce from the above that if V and W are two simple representations of G then $V \cong W$ if and only if $\chi_V = \chi_W$.

(3) Determine the indecomposable projectives and their radicals for the following bound quivers.
(a)

$$\bullet_1 \xrightarrow{\alpha_1} \bullet_2 \xrightarrow{\alpha_2} \bullet_3 \xrightarrow{\alpha_3} \bullet_4$$

(b) The same quiver as in (a) but with the relation $\alpha_3\alpha_2 = 0$.

(c)

$$\begin{tikzcd} \bullet_1 \arrow[r, "\alpha_1"] \arrow[d, "\alpha_3"] & \bullet_2 \arrow[loop right, "\beta"] \arrow[d, "\alpha_2"] \\ \bullet_3 \arrow[r, "\alpha_4"] & \bullet_4 \end{tikzcd}$$

with the relation

$$\beta^2 = 0.$$

(d) The same quiver as in (c) with the relations

$$\beta^2 = 0 \qquad \alpha_2\alpha_1 = 2\alpha_4\alpha_3.$$

(4) Suppose that Q is a finite acyclic quiver. Show that all the linear maps in an indecomposable projective representation of Q must be injective.