CHAPTER 5

Representation type and Gabriel's theorem

5.1. Representation type

We have seen that a finite dimensional algebra has only finitely many isomorphism classes of simple modules. It is natural to ask if the same is also true of indecomposables. However, this is not generally the case.

DEFINITION 5.1.1. *An algebra has* finite representation type *if there are only finitely many isomorphism classes of finite dimensional indecomposable modules. Otherwise the algebra has* infinite representation type.

By Krull-Schmidt it is clear that for a representation finite algebra we have complete knowledge of its representation theory once we have constructed a complete (finite) set of indecomposable modules (although that is not necessarily easy!). Semisimple algebras are clearly of finite representation type.

EXAMPLE 5.1.2. *Suppose that k is algebraically closed. Then the algebra $A = k[x]/(x^n)$ has finite representation type. Any A-module M is a vector space together with a linear map $\phi : M \longrightarrow M$ such that $\phi^n = 0$. Consider ϕ as a matrix with respect to some basis. Then the corresponding Jordan canonical form for ϕ is a block diagonal matrix where each block is a $t \times t$ matrix of the form*

$$J_t(0) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

for some $t \leq n$ (as no larger block satisfies $\phi^n = 0$). But if M is indecomposable then there is only one such block. Therefore there are precisely n isomorphism classes of indecomposable modules: one each of dimension $1, 2, \ldots, n$.

EXAMPLE 5.1.3. *The algebra $A = k[x,y]/(x^2, y^2)$ has infinite representation type. Let $M = k^{2n}$ for some $n \geq 1$ and chose $\lambda \in k$. Then let x and y act respectively by*

$$X = \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & J_n(\lambda) \\ 0 & 0 \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix, and $J_n(\lambda) = J_n(0) + \lambda I_n$. It is easy to verify that $X^2 = Y^2 = 0$ and $XY = YX$, so this defines an A-module. One can also check it is indecomposable. Clearly these modules are non-isomorphic for different values of n (and in fact they are also non-isomorphic for different values of λ).

EXAMPLE 5.1.4. *Let Q be a quiver such that there exist two vertices i and j such that there are (at least) two arrows $\alpha, \beta : i \longrightarrow j$. Then as in the last example we find infinitely many non-isomorphic indecomposables by setting $M_i = M_j = k^n$ and representing α by the matrix I_n and β by the matrix $J_n(\lambda)$.*

LEMMA 5.1.5. *If A has finite representation type and I is an ideal in A then A/I has finite representation type.*

PROOF. Suppose that M is an A/I-module. Then we can define an A-module structure on M by setting $am = (a+I)m$ for all $a \in A$ and $m \in M$. Further M is indecomposable for A if and only if it is for A/I, and two modules are isomorphic as A/I-modules if and only they are isomorphic as A-modules. □

If an algebra is not representation finite, is there any hope to classify the finite dimensional indecomposable modules?

DEFINITION 5.1.6. *Suppose that k is an infinite field. An algebra over k has* tame representation type *if it is of infinite type and for all $n \in \mathbb{N}$, all but finitely many isomorphism classes of n-dimensional indecomposables occur in a finite number of one-parameter families.*

Thus there is some hope that one can classify all indecomposable representations for algebras of tame representation type.

REMARK 5.1.7. *(a) We could make precise what we mean by a one-parameter family of representations; for our purposes however the above definition will be good enough. The idea of a one-parameter family is illustrated in the variation with λ of the representations defined in Example 5.1.3.*
(b) Some authors define tame representation type to include finite representation type.

DEFINITION 5.1.8. *A k-algebra A has* wild representation type *if for all finite dimensional k-algebras B, the representation theory of B can be embedded into that of A.*

REMARK 5.1.9. *Again, we could give a more precise definition of what we mean by embedding one representation theory inside another, but this would require the language of categories.*

This means that understanding all indecomposable representations of A implies an understanding of all representations of *every* finite dimensional algebra, which should sound like a hopeless task. That it is can be seem from

REMARK 5.1.10. *It follows from an alternative definition of wild representation type that the representation theory of $k\langle x,y\rangle$ can be embedded into that of any wild algebra. But the word problem for finitely presented groups can be embedded into the representation theory of $k\langle x,y\rangle$, and this problem has been proved to be* undecidable.

The following fundamental theorem is due to Drozd.

THEOREM 5.1.11 (Trichotomy theorem). *Over an algebraically closed field, every finite dimensional algebra is either of finite, tame, or wild representation type.*

PROOF. A proof of this theorem is beyond the scope of this course. □

In general we do not have a complete classification of algebras of finite (or tame) representation type. However in the special case of a quiver algebra or a group algebra we can give such a classification. We will conclude this section by considering the group case. As one would expect from Maschke's theorem, this now depends on the field as well as the group. We begin with a special case.

PROPOSITION 5.1.12. *Let G be a finite group of order p^n, and k be a field of characteristic p. Then kG has finite representation type if and only if G is cyclic.*

PROOF. (Sketch) First suppose that G is cyclic. Then by Example 5.1.2 it is enough to show that $kG \cong k[x]/(x^{p^n})$. Let g be a generator for G, and define a map $\phi : k[x] \longrightarrow kG$ by $f \longmapsto f(1-g)$.

We claim that this is a surjective algebra homomorphism, with kernel containing (x^{p^n}). From this it follows by comparing dimensions and Lemma 1.2.7 that ϕ induces the desired algebra isomorphism. To see the claim, note that $(1-g)^{p^i} = 1-g^{p^i}$ in characteristic p, as all other binomial coefficients vanish, and hence $\phi(x^{p^n}) = 0$. Then verify that $1, (1-g), (1-g)^2, \ldots, (1-g)^{p^n-1}$ form a basis for kG.

For the reverse implication, basic group theory implies that there exists $N \lhd G$ such that $G/N \cong C_p \times C_p$. It is then enough by Lemma 5.1.5 to show that $k(G/N)$ has infinite representation type. By a similar argument to the preceding paragraph, one can show that

$$k(G/N) \cong k[x,y]/(x^p, y^p).$$

As $(x^p, y^p) \subseteq (x^2, y^2)$, it is enough to show that $k[x,y]/(x^2,y^2)$ has infinite type. But this was done in Example 5.1.3. □

Using this it is possible to prove

THEOREM 5.1.13 (Higman). *Let G be a finite group and k a field. Then kG has finite representation type if and only if either*
(a) k has characteristic zero, or
(b) k has characteristic $p > 0$ and G has a cyclic Sylow p-subgroup.

PROOF. (Sketch) If k has characteristic zero then kG is semisimple by Maschke's theorem, and we are done. If k has positive characteristic then we would like to argue that kG of finite type if and only if kH is of finite type where H is a Sylow p-subgroup of G, as then we are done by Proposition 5.1.12.

As H is a Sylow p-subgroup of G the index of H in G is coprime to p, and so is non-zero in k. The reduction to the case of kH now proceeds by a Maschke-type averaging argument. □

The tame cases can also be classified.

THEOREM 5.1.14. *Let G be a finite group, and k be an infinite field of characteristic $p > 0$. Then kG has tame representation type if and only if $p = 2$ and the Sylow 2-subgroups of G are dihedral, semidihedral, or generalised quaternion.*

5.2. Representation type of quiver algebras

In the special case of a quiver algebra we have a complete classification of those of finite and of tame representation types. We will begin by consider the finite type case, for which we will need to introduce certain Dynkin diagrams. These are illustrated in Figure 5.1.

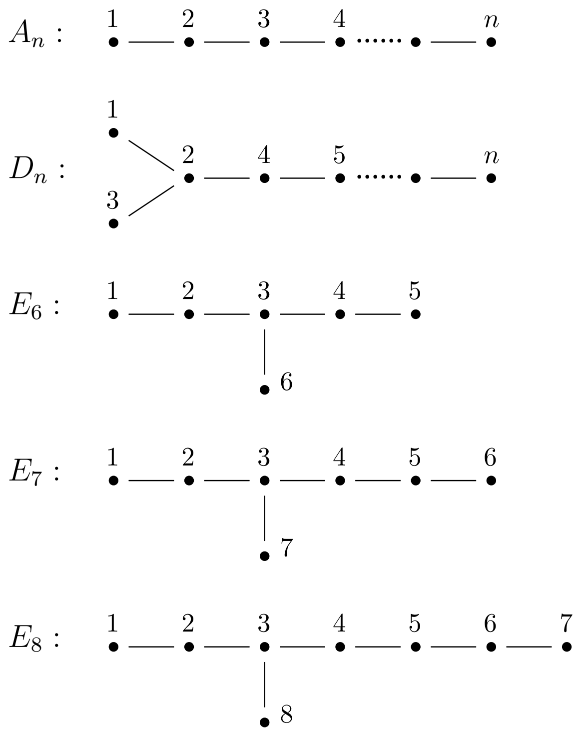

FIGURE 5.1. The Dynkin diagrams of types A_n, D_n, E_6, E_7, and E_8

THEOREM 5.2.1 (Gabriel). *Suppose that Q is a finite quiver. Then kQ has finite representation type if and only if $\underline{Q}$ is a disjoint union of Dynkin diagrams of types A, D, or E as in Figure 5.1.*

If you know any of the theory of Lie algebras then you may recognise Dynkin diagrams as being associated with a root system. (This explains the strange labelling scheme: there are also root systems of types B_n and C_n, as well as F_4 and G_2.)

There is a similar classification of tame quiver algebras, this time in terms of certain *extended Dynkin diagrams* (also known as *Euclidean diagrams*).

THEOREM 5.2.2. *Suppose that Q is a finite quiver and k an infinite field. Then kQ has tame representation type if and only if $\underline{Q}$ is a disjoint union of extended Dynkin diagrams as in Figure 5.2 possibly together with Dynkin diagrams of types A, D, E as in Figure 5.1.*

In the next two sections we will introduce some of the main ideas used in the proof of Gabriel's theorem. First we will introduce some combinatorics associated to representations which for simples and projectives only depends on the underlying graph. This provides the link with the language of Lie theory (although a knowledge of this is not necessary here). In the final section of this chapter we will outline how this combinatorics, together with reflection functors, can be used to prove one implication of Gabriel's theorem.

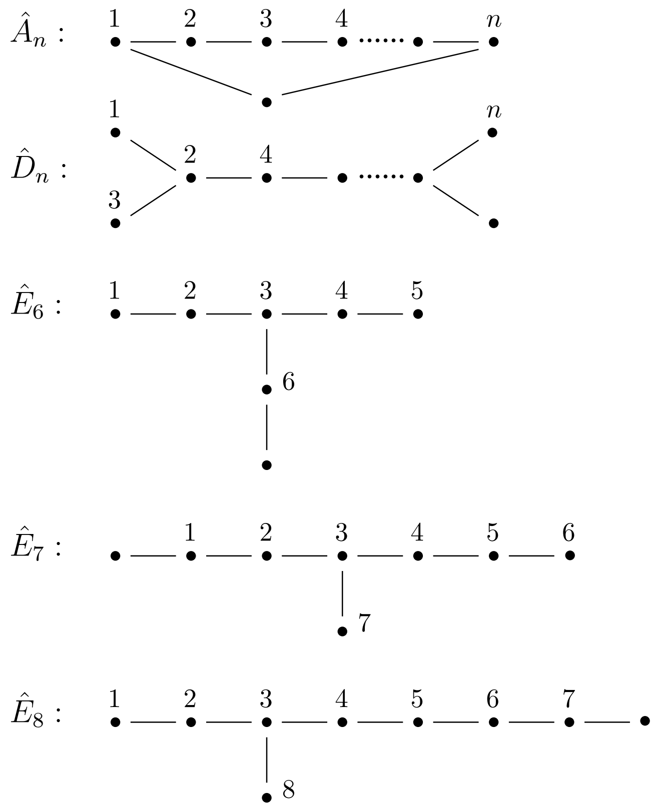

FIGURE 5.2. The extended Dynkin diagrams of types $\hat{A}_n$, $\hat{D}_n$, $\hat{E}_6$, $\hat{E}_7$, and $\hat{E}_8$

5.3. Dimension vectors and Cartan matrices

In this section we will assume for convenience that the vertex set Q_0 of a finite quiver Q has been identified with $\{1,\dots,n\}$ for some n.

DEFINITION 5.3.1. *Suppose that $M = (M_i, \phi_i)$ is a representation of a finite quiver Q with vertices $1,\dots n$. Then the* dimension vector *of M is the n-tuple*

$$\underline{\dim} M = (\dim M_1, \dots, \dim M_n).$$

EXAMPLE 5.3.2. *(a) The dimension vector of the representation considered in Example 1.4.3 is $(1,2,2,3,2)$.*

(b) Clearly the simple representations of Q have dimension vectors with 1 in the ith position (for some i) and 0 elsewhere. We will denote this vector by $e(i)$.

(c) By Proposition 3.3.2 we have that $p(i) = \underline{\dim} P(i)$ is the vector whose jth coordinate is the number of paths from i to j.

We can now define a matrix related to kQ which will play an important role in what follows.

DEFINITION 5.3.3. *The* Cartan matrix *C of kQ is the $n \times n$ matrix whose ith column is the vector $p(i)^T$.*

EXAMPLE 5.3.4. *Let Q be the quiver*

$$\bullet_1 \xrightarrow{\alpha} \bullet_2 \xrightarrow{\beta} \bullet_3 \xrightarrow{\gamma} \bullet_4 \; .$$

This has Cartan matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

LEMMA 5.3.5. *For all $i \in Q_0$ we have*

$$e(i) = p(i) - \sum_{i=1}^{n} a(i,j)p(j) \tag{6}$$

where $a(i,j)$ is the number of arrows from i to j.

PROOF. (Sketch) Let $A = kQ$, and set $S_i = Ae_i/\mathscr{J} e_i$. Then we have that $e(i) = \underline{\dim}(S_i)$ and $p(i) = \underline{\dim}(Ae_i)$ and so

$$e(i) = \underline{\dim}(Ae_i) - \underline{\dim}(\mathscr{J} e_i).$$

Thus it is enough to show that

$$\underline{\dim}(\mathscr{J} e_i) = \sum_{i=1}^{n} a(i,j)p(j).$$

Now $\mathscr{J} e_i$ is the span of all paths of positive length starting at i, which equals the direct sum of all $A\alpha$ where α is an arrow starting at i. It is easy to see that $A\alpha \cong A\varepsilon_j$ where $\alpha : i \longrightarrow j$ via the map $x\alpha \longmapsto x\varepsilon_j$. □

COROLLARY 5.3.6. *The Cartan matrix of Q is invertible over $\mathbb{Z}$.*

PROOF. Transposing the vectors in (6) we obtain

$$e(i)^T = p(i)^T - \sum_{i=1}^{n} a(i,j)p(j)^T.$$

The left-hand side is the columns of the identity matrix, while the right-hand side involves the columns of C. Thus C has a left inverse $I + (-a(i,j))$. □

EXAMPLE 5.3.7. *Returning to the quiver in Example 5.3.4 we see that*

$$C^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

We can now use the Cartan matrix to define various forms on $\mathbb{Z}^n$. We will write C^{-T} for $(C^{-1})^T$.

DEFINITION 5.3.8. *We define the* Euler characteristic, *a (not in general symmetric) bilinear form on $\mathbb{Z}^n$ by*

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} C^{-T} \mathbf{y}^T$$

and an associated symmetric form by

$$(\mathbf{x},\mathbf{y}) = \langle \mathbf{x},\mathbf{y}\rangle + \langle \mathbf{y},\mathbf{x}\rangle.$$

It is an elementary exercise to show that

LEMMA 5.3.9. *For all i and j in Q_0 we have*

$$\langle p(i), e(j)\rangle = \delta_{ij}.$$

DEFINITION 5.3.10. *For $i \in Q_0$ define a map $s_i : \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$ by*

$$s_i(\mathbf{x}) = \mathbf{x} - (\mathbf{x}, e(i))e(i).$$

This is a linear map and it is easy to verify that $s_i^2 = \mathrm{id}$. We define W to be the subgroup of $\mathrm{GL}_n(\mathbb{Z})$ generated by the s_i. We say that $\mathbf{x} \in \mathbb{Z}^n$ is positive *if $x_i \geq 0$ for all i, with strict inequality for at least one i, and write $\mathbf{x} > 0$. Then the set of* positive roots *for Q is the set*

$$\{w(e(i)) : w(e(i)) > 0, \quad w \in W, \quad 1 \leq i \leq n\}.$$

Root systems arise in a variety of places, such as Lie theory, and are well understood. The following fact is not hard to prove.

LEMMA 5.3.11. *Suppose that $\overline{Q}$ is of type A, D, or E. Then the set*

$$\{w(e(i)) : \quad w \in W, \quad 1 \leq i \leq n\}$$

(and hence the set of positive roots) is finite.

The relevance of the above combinatorial framework to representation theory is the following result.

THEOREM 5.3.12 (Gabriel). *Suppose that Q is a finite quiver such that $\overline{Q}$ is of type A, D or E. Then the map $V \longmapsto \underline{\dim} V$ gives a bijection between isomorphism classes of finite dimensional indecomposable representations and the positive roots of Q.*

(Combining this with Lemma 5.3.11 proves that ADE type quivers have finite representation type.)

One way to prove Theorem 5.3.12 is using reflection functors.

5.4. Reflection functors

DEFINITION 5.4.1. *Let Q be a finite quiver, and suppose that i is a vertex such that there are no arrows starting from i. Then we say that i is a* sink *in Q. Similarly, if there are no arrows ending at i then we say that i is a* source *in Q.*

Suppose that i is a sink (or source) of Q. We wish to define a new quiver s_iQ and a functor from kQ-modules to ks_iQ-modules. (This just means that it should map kQ-modules to ks_iQ-modules and also map morphisms between kQ-modules to corresponding morphisms for ks_iQ in a compatible manner.) We begin with s_iQ.

DEFINITION 5.4.2. *Suppose that i is a sink (or source) of Q. Then s_iQ is the quiver obtained by reversing the direction of all arrows ending at i.*

Now suppose that $M = (M_i, \phi_i)$ is a representation of Q. We next wish to define a representation of s_iQ when i is a sink. Suppose for concreteness that the arrows entering i are labelled α_j with $\alpha_j : i_j \longrightarrow i$ for $1 \leq j \leq t$.

DEFINITION 5.4.3. *Let $C_i^+(M)$ be the s_iQ-representation with $C^+(M)_j = M_j$ for all $j \neq i$. The space $C_i^+(M)_i$ is defined by the exact sequence*

$$0 \longrightarrow C_i^+(M)_i \xrightarrow{\theta} \bigoplus_{j=1}^{t} M_{i_j} \xrightarrow{\psi} M_i \tag{7}$$

where $\psi = \sum_{j=1}^{t} \phi_{i_j}$. The linear maps in $C_i^+(M)$ are unchanged if the arrow has not been reversed, and are θ followed by projection onto the relevant summand if the arrow has been reversed. Given a morphism ϕ between two representations of Q a corresponding morphism $C_i^+(\phi)$ can be defined between s_iQ-modules, which makes C_i^+ into a functor. We call this a reflection functor.

As the notation suggests, there is a relation between reflection functors and the combinatorics of the preceding section. This follows from

PROPOSITION 5.4.4. *Suppose that M is a finite dimensional indecomposable representation of a finite quiver Q. Then $C_i^+(M)$ is 0 if M is a simple representation, and is indecomposable otherwise. In the latter case we have that*

$$\underline{\dim} C_i^+(M) = s_i(\underline{\dim} M).$$

PROOF. (Sketch) It is clear that $C_i^+(M) = 0$ if M is simple. Next one shows: (i) that M is indecomposable only if M is simple or the map ψ in (7) is surjective, and (ii) that if $N = C_i^+(M)$ then there is a homomorphism

$$\mathrm{End}_{kQ}(M) \longrightarrow \mathrm{End}_{ks_iQ}(N)$$

which is surjective if (7) is surjective.

Now suppose M is indecomposable but not simple. Then $\mathrm{End}_{kQ}(M)$ is local by Lemma 2.4.6 and we have a surjection onto $\mathrm{End}_{ks_iQ}(N)$. Arguing as in the proof of 3.1.7 we see that this latter algebra is also local, and so N is indecomposable.

The dimension claim follows from elementary linear algebra, together with a comparison with the corresponding combinatorics for dimension vectors. □

Now suppose that i is a source in Q. There is a similar definition of a reflection functor C_i^- in this case were we reverse the direction of all the arrows in (7). Again one can show that C_i^- takes simple representations to 0 and non-simple indecomposable to indecomposables as in Proposition 5.4.4. From the definitions it is easy to verify that

$$C_i^- C_i^+(M) \cong \begin{cases} M & M \ncong S_i \\ 0 & M \cong S_i \end{cases}$$

and similarly

$$C_i^+ C_i^-(M) \cong \begin{cases} M & M \ncong S_i \\ 0 & M \cong S_i. \end{cases}$$

From this follows

COROLLARY 5.4.5. *Suppose that i is a sink in Q. Then there is a bijection between non-simple finite dimensional indecomposable kQ-modules and non-simple indecomposable ks_iQ-modules given by C_i^+. Hence kQ and ks_iQ have the same representation type.*

Any finite acyclic quiver has a sink and a source. Thus we can label the vertices of Q starting with the sinks, then taking the sinks in the quiver without these vertices, and so on. Thus we may assume that if there is an edge from i to j then $i < j$. We will call such a labelling an *admissible labelling*.

DEFINITION 5.4.6. *Suppose that Q has an admissible labelling. Then the functor*

$$C^+ = C_n^+ C_{n-1}^+ \dots C_1^+$$

is defined. We call this the Coxeter functor *with respect to this ordering. Note that every arrow in Q is reversed precisely twice in the construction of C^+, and so C^+ takes representations of Q to representations of Q. Similarly there is a functor $C^- = C_1^- \dots C_n^-$. There are corresponding elements s^+ and s^- in W.*

Using the finiteness of the set of positive roots from Lemma 5.3.11 it is now possible to prove

LEMMA 5.4.7. *If $\mathbf{y} \in \mathbb{Z}^n$ satisfies $s^+\mathbf{y} = \mathbf{y}$ then $\mathbf{y} = 0$. Also, if $\mathbf{x} \in \mathbb{Z}^n$ with $\mathbf{x} > 0$ then $(s^+)^n\mathbf{x} = 0$ for $n >> 0$.*

Now we can sketch the proof of Theorem 5.3.12.

PROOF. (Sketch) First suppose that Q is of type ADE, and that M is a finite dimensional indecomposable representation of Q. Then for $n >> 0$ we have $(C^+)^n M = 0$. This follows from Lemma 5.4.7 as $\underline{\dim}(C^+)^n M \geq 0$ for all n, but equals $(s^+)^n \underline{\dim} M$ by Proposition 5.4.4.

Thus there exists n such that $X = (C^+)^n M \neq 0$ but $(C^+)X = 0$. Therefore there is an i such that $C_{i-1}^+ \dots C_1^+(X) \neq 0$ but $C_i^+ C_{i-1}^+ \dots C_1^+(X) = 0$. By Proposition 5.4.4 this implies that $C_i^+ \dots C_1^+(X) \cong S_i$ (for the relevant quiver). We can reverse our steps and reconstruct M from S_i using C_j^- functors, which also gives the dimension vector for M in terms of the action of W on $e(i)$. It is easy to see that this gives the desired bijection between dimension vectors and finite dimensional indecomposable modules. ☐

This gives one half of Gabriel's Theorem 5.2.1. To prove that all other quivers have infinite representation type, one proceeds case by case. Show that various simple quivers have infinite type by hand, such as a quiver with multiple arrows (see Example 5.1.4), or a quiver with four arrows from distinct vertices meeting at a single vertex. Then show that every quiver contains one of these examples as a subquiver (and hence is of infinite type) except the quivers of type ADE. Finally by using reflection functors we see that the representation type depends only on the underlying graph.

To conclude, a word or two about infinite dimensional representations. As one might expect these are considerably more complicated. Here are two general theorems for the finite and infinite type cases.

THEOREM 5.4.8 (Auslander). *If A is a finite dimensional algebra of finite representation type then every indecomposable A-module is finite dimensional, and every module is a direct sum of indecomposables.*

THEOREM 5.4.9 (Roiter). *If A is a finite dimensional algebra of infinite representation type then there are indecomposable A-modules with arbitrarily many composition factors.*

5.5. Exercises

(1) Let Q be the quiver

$$\begin{tikzcd} \bullet_1 \arrow[r, "\alpha"] & \bullet_2 & \bullet_3 \arrow[l, "\beta"'] \end{tikzcd} .$$

  (a) Show that this has six isomorphism classes of indecomposable modules with dimension vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(0,1,1)$ and $(1,1,1)$.
  (b) Determine the Cartan matrix for Q.
  (c) Verify that the dimension vectors of projective and simple representations are orthogonal with respect to the Euler characteristic.
  (d) Determine the group W as a subgroup of $\mathrm{GL}_3(\mathbb{Z})$, and hence verify that the dimension vectors of indecomposable modules are in bijection with the positive roots of Q.
  (e) Consider the reflection functor for Q corresponding to the unique sink in Q. Determine the effect of this functor on each of the indecomposable representations of Q, verifying that in each case the new representation is indecomposable.

(2) Consider the 3-*Kronecker quiver* Q given by

$$\begin{tikzcd} \bullet_1 \arrow[r, bend left, "\alpha"] \arrow[r, "\beta"] \arrow[r, bend right, "\gamma"'] & \bullet_2 \end{tikzcd} .$$

For $\lambda, \mu \in k$ let $M(\lambda,\mu)$ be the representation of Q such that $M_1 = k^2$, $M_2 = k$, and the maps corresponding to α, β and γ are given by the matrices $(1,0)$, $(0,1)$, and (λ,μ) respectively. Show that the representations $M(\lambda,\mu)$ are indecomposable and pairwise non-isomorphic (and hence that this quiver has a two-parameter family of indecomposables).

These are not the only indecomposables (this quiver has wild representation type!). In [**Bar06**, Proposition 2.1] it is shown that classifying the indecomposables of this quiver would allow one to classify the indecomposables for *any* quiver.

(3) Investigate what happens if you apply the theory of reflection functors to the 3-Kronecker quiver and its representations $M(\lambda,\mu)$ described above.

CHAPTER 6

Further directions

In this Chapter we will briefly review some of the many ways in which the material in this course can be extended. Given the time available we can only sketch an indication of the kind of topics that can be covered: more detailed surveys can be found in the references.

6.1. Ring theory

Much of the classical material developed in Chapters 1-3 can also be considered when the field k is replaced by a (commutative) ring. However this can introduce considerable complications — particularly when we consider representations over the integers. Good basic introductions can be found in [**Mat86**] and the (194 page!) introduction to [**CR81**]. The latter also gives an extensive exposition of the integral representation theory of finite groups. A shorter discussion more in the spirit of the later part of these notes can be found in [**Ben91**].

6.2. Almost split sequences and the geometry of representations

We have only begun the study of representations of finite dimensional algebras. There are several important ideas which we have not had time to touch on, and we will give a brief sketch of a few of them in this section.

Consider a short exact sequence of A-modules

$$0 \to L \xrightarrow{\phi} M \xrightarrow{\psi} N \to 0.$$

If M is the direct sum of L and N then we call the sequence *split*. Recall from Lemma 3.1.2 that this is equvalent to the existence of a left inverse to ϕ and to the existence of a right inverse to ψ. We call a morphism with a left inverse a *section*, and with a right inverse a *retraction*.

Clearly if our sequence is split we understand M completely if we understand L and N. However, we would like to be able to deal with non-split sequences. Almost split sequences turn out to play an important role.

We say that a homomorphism $\phi : L \to M$ is *left minimal* if every elements $\theta \in \mathrm{End}_A(M)$ with $\theta\phi = \phi$ is an automorphism. (There is a similar definition for *right minimal*.) The map ϕ as above is called *left almost split* if ϕ is not a section, and for every morphism $\tau : L \to U$ that is not a section there exists $\tau' : M \to U$ such that $\tau'\phi = \tau$. This definition is similar to that for an injective module; the corresponding 'projective' version is called *right almost split*.

Now we can give the main definition. A sequence

$$0 \to L \xrightarrow{\phi} M \xrightarrow{\psi} N \to 0$$

is *almost split* if ϕ is left minimal and left almost split, and ψ is right minimal and right almost split. It is clear that an almost split sequence is not split. However, it is not immediately clear that there exist any such sequences.

First one shows that if $\phi : L \to M$ is left minimal and left almost split, then M is unique up to isomorphism. If ϕ is merely left almost split then L must be an indecomposable. (And of course there are similar righthand versions of these results.) Thus if there is an almost split sequence as above then L and N must be indecomposable, and M is uniquely determined. Further, L cannot be injective and N cannot be projective.

The *Auslander-Reiten translate* is an explicit functor which takes an A-module M to an A-module τM. (The precise definition is a little too involved for the time available to us.) Using this, Auslander and Reiten were able to prove

THEOREM 6.2.1 (Auslander-Reiten). *If M is indecomposable and not projective then there is exists an almost split sequence*

$$0 \to \tau M \to E \to M \to 0.$$

There is a similar result for indecomposable noninjectives using the inverse translate τ^{-1}.

Auslander and Reiten also introduced the *Auslander-Reiten quiver* associated to a finite dimensional algebra A. This is a quiver whose vertices are the isomorphism classes of indecomposable representations of A, and whose arrows correspond to bases for the spaces of certain *irreducible morphisms* between indecomposables. Studying this, together with the effect of the Auslander-Reiten translate upon it, is an important aspect of the modern theory.

For example, using this one can prove the following conjectures of Brauer and Thrall:

THEOREM 6.2.2. *If A is not representation finite then A has indecomposable modules of arbitrarily large dimension.*

THEOREM 6.2.3. *If k is algebraically closed and A is not representation finite then there are infinitely many positive integers n such that there are infinitely many non-isomorphic n-dimensional indecomposable A-modules.*

The theory of almost split sequences and AR-quivers is developed in [**ARS94**] and [**ASS06**].

Another direction of study is inspired by the reflection functors used in the proof of Gabriel's Theorem. This leads to a general area of *tilting theory*, which tries to replace the algebra A being studied by a simpler algebra which is closely related. Again, an extensive theory has been developed — see for example [**ASS06**].

Finally in this section, we should note that there is an important approach to representations of finite dimensional algebras which we have completely ignored in these notes, which relies on geometric techniques.

If we fix a dimension vector α, then the space of representations of a given quiver with that dimension forms an algebraic variety. Thus we may use the methods of algebraic geometry to study this variety. This is a very powerful technique, but does require more geometry than we have time to introduce in these notes. For an indication of how the results in this course (such as Gabriel's theorem) can be approached in this manner, see [**Bru03**]. There is also a more general survey focussing on the geometric aspects in [**CB93**].

6.3. Local representation theory

We have not looked in detail at the special case of group algebras of finite groups in characteristic $p > 0$. We did see in the discussion of representation type that the Sylow p-subgroups play a key role. There is a general approach to studying group representations which proceeds by relating the representation theory of a group G to that of certain normalisers of p-groups in G.

Given $H \leq G$, we can generalise the notion of projective modules for G to *relatively H-projective* modules. One way to define this is to copy the definition we have given, but add the requirement that the desired homomorphism must exist as a morphism of kH-modules. Using this the Green and Brauer correspondences can be defined which reduce to the study of the representation theory of normalisers of p-groups in G.

This leads to an extensive and well-developed theory. An excellent introduction, which starts in the spirit of these notes, can be found in [**Alp86**].

6.4. Representations of other algebraic objects

In this series of lecures we have concentrated on representations of (mainly finite dimensional) associative algebras. But there are other algebraic structures we could have studied. We will introduce a few of the most important examples.

A *Lie algebra* is an example of a non-associative algebra. The bilinear map of two elements x and y is traditionally denoted by $[x,y]$. To give a Lie algebra structure this map must be *antisymmetric*:

$$[x,x] = 0$$

and satisfy the *Jacobi identity*:

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.$$

Given two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, a *homomorphism* from $\mathfrak{g}$ to $\mathfrak{h}$ is a linear map which respects the Lie algebra structures, i.e. such that

$$\phi([x,y]) = [\phi(x), \phi(y)].$$

Note that any associative algebra A can be given a Lie algebra structure by using the standard multiplication to define

$$[x,y] = xy - yx.$$

In particular, given a vector space M the algebra $\mathrm{End}_k(M)$ has a Lie algebra structure, and we define a *representation* of $\mathfrak{g}$ to be a vector space M together with a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathrm{End}_k(M)$.

In a similar way we can define representations of various other classes of algebraic objects by showing that $\mathrm{End}_k(M)$ or $\mathrm{Aut}_k(M)$ (the space of invertible linear maps) lies in that class, and requiring that the linear map is a homomorphism in that class.

For example, if $k = \mathbb{C}$ or $\mathbb{R}$ and we start with a *Lie group* G (a group that is also a differentiable manifold, such that the group operations are smooth maps) then if M is a finite dimensional vector space then $\mathrm{Aut}_k(M)$ is also a Lie group. (For infinite dimensional M more care is needed.) Thus a *representation* of G is a vector space M and a homomorphism of Lie groups from G to $\mathrm{Aut}_k(M)$. This situation can be generalised to arbitrary algebraically closed fields by considering *algebraic* *groups* and their representations. Instead of being a differential manifold we require that the group is an algebraic variety with group operations which are morphisms of varieties.

The representation theory of Lie algebras and of Lie or algebraic groups is closely related, and all three theories have been very well developed. An introduction to the basics of Lie algebra representation theory can be found in [**Car05**] or [**Hum72**]. For Lie groups it is necessary to know some basic manifold theory, while for algebraic groups there is a fair amount of algebraic geometry required. See [**FH91**] (or the more advanced [**Bum04**]) for Lie groups and [**Jan03**] for algebraic groups — although the latter presumes a good knowledge of the basic structure of such groups as described in [**Spr98**] or [**Hum75**].

Given a Lie algebra $\mathfrak{g}$, there exists a corresponding *universal enveloping algebras* $U(\mathfrak{g})$. This is an infinite dimensional associative algebra which (via the usual Lie algebra structure on an associative algebra) preserves the representation theory of $\mathfrak{g}$. A classical introduction is [**Dix96**]; see also [**Hum08**] for a guide to the related *category* $\mathscr{O}$ of certain infinite dimensional representations of $\mathfrak{g}$ over $\mathbb{C}$.

The special class of *semisimple algebraic groups* (or the associated Lie algebras) can be classified; the classification is based around Dynkin diagrams. There are corresponding *finite groups of Lie type*, and one way of studying these is via a reduction from the corresponding algebraic group. An introduction to this approach can be found in [**DM91**], while [**Hum06**] gives a more elementary and up-to-date survey of the field.

6.5. Quantum groups and the Ringel-Hall algebra

To define an algebra we needed a multiplication map: a bilinear map from $A\times A$ to A. We can define an analogous structure called a *coalgebra* by defining every map in the opposite direction, and considering *comultiplication*: a bilinear map from A to $A\times A$. (There are various conditions in the definition which we will not describe here.) Algebras that are also coalgebras in a compatible way are called *bialgebras*, and if they have one additional property (corresponding to the inversion of elements in a group) we obtain a *Hopf algebra*. There are plenty of interesting examples of Hopf algebras — including group algebras and the universal enveloping algebra of a Lie algebra.

Quantum groups have been defined in a number of different ways. In each case, the basic idea is to take some Hopf algebra related to a Lie algebra $\mathfrak{g}$ and introduce an extra parameter $q\in k$. The structure of these algebras will depend on q, but when q tends to 1 we should recover the original Hopf algebra in the limit. The standard construction is realised as a deformation of the universal enveloping algebra of $\mathfrak{g}$.

Quantum groups have been studied for many reasons. They arose in the mathematical physics literature (which is a rich source of interesting representation theories), and have since proved very useful in the study of representations of algebraic groups in positive characteristic. (The best results to date on the structure of simple modules for algebraic groups proceed via a comparison with the associated quantum group.) They have also shed new light on the classical theory; certain remarkable bases called crystal (or canonical) bases were found in the quantum world which were not previously known in the classical case.

There are many different approaches to quantum groups, reflecting their varied applications. Two good examples are [**Jan96**] and [**Kas95**]. There is also a nice introduction to the theory of crystal bases in [**HK02**].

Why have we made a detour into Lie theory in the last two sections? Well, it turns out that quantum groups are closely related to representations of finite dimensional algebras. Ringel (generalising work of Hall) defined certain algebras, the *Ringel-Hall algebras* associated to a finite dimensional algebra A. These have basis the set of isomorphism classes of indecomposable A-modules, and multiplication is defined in terms of possible extensions of one module by another. Ringel then proved that for a quiver algebra this algebra is isomorphic to a quantum group associated to the corresponding Lie algebra. Thus the theory of finite dimensional algebras is closely related to that of Lie algebras. The relationship between these two theories is described in [**DDPW08**].

6.6. Categorification and higher representation theory

Categorification is the process whereby a set-theoretic structure is enriched into a category-theoretic structure. In this process, each set is replaced by a category, with functions replaced by functors and equations holding in the structure by natural isomorphisms of functors which are themselves related by further equations. One rather elementary example of a categorification is the relation between the natural numbers $\mathbb{N}$ and the category of finite sets.

Indeed, this process can be extended to categories themselves, to form the notion of higher categories. For example a 2-category will consists of objects, morphisms between objects, and 2-morphisms between morphisms. This process is in part motivated by problems and ideas in homotopy theory. An introduction to the general notions can be found in [**BD98**].

Categorifaction has had a number of very striking applications in representation theory. A survey of some of these can be found in [**Maz12**]. The most famous of these is probably the proof of Broué's Abelian Defect Group Conjecture for the symmetric groups by Chuang and Rouquier. The key idea in this work was to realise the complexification of the character ring of the group algebra of the symmetric group as the basic highest weight representation of some affine Kac-Moody Lie algebra. This work has been extended and generalised by Rouquier, and by Khovanov and Lauda, into a more general notion of higher representation theory.

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