Linear Algebra Answers, Summer 2003

Note: these are only the final answers. In an exam you must also give full working and explanations where required. In some cases alternative choices are possible (for example for some of the bases).

Question 5:

(a) Solutions of $\det(A-\lambda I)=0$ are $\lambda = 3, 5$. **[3 marks]**

For $\lambda = 5$ a basis is given by $\begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$. **[3 marks]**

For $\lambda = 3$ the eigenspace consists of vectors of the form $\begin{pmatrix} x \\ y \\ x+y \end{pmatrix}$. A basis is given by $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Need to check linearly independent and spanning. **[7 marks]**

Choose $P = \begin{pmatrix} -1 & 1 & 0 \\ 4 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}$, then $P^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & -1 \\ -4 & -3 & 4 \end{pmatrix}$ and $D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.
[3 marks]

(b) Cayley-Hamilton Theorem: see notes.

$p_A(\lambda) = -\lambda^3 + 11\lambda^2 - 39\lambda + 45.$

$$\begin{aligned} p_A(A) &= -A^3 + 11A^2 - 39A + 45I \\ &= \begin{pmatrix} 71 & 98 & -98 \\ -392 & -419 & 392 \\ -196 & -196 & 169 \end{pmatrix} + \begin{pmatrix} -77 & -176 & 176 \\ 704 & 803 & -704 \\ 352 & 352 & -253 \end{pmatrix} - \begin{pmatrix} 39 & -78 & 78 \\ 312 & 429 & -312 \\ 156 & 156 & -39 \end{pmatrix} \\ &\qquad + \begin{pmatrix} 45 & 0 & 0 \\ 0 & 45 & 0 \\ 0 & 0 & 45 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

[4 marks]

Question 6:

(a)(i) Not a subspace, e.g. $\mathbf{x} = (1,0,\dots,0)$ and $\mathbf{y} = (0,0,\dots,1)$ are in A but not $\mathbf{x}+\mathbf{y}$.

(ii) This is a subspace.

(iii) This is not a subspace, e.g. as $\mathbf{0}$ is not in C. **[6 marks]**

(b)(i) Linearly independent but does not span.

(ii) Is a basis.

(iii) Neither linearly independent nor spanning. **[10 marks]**

(c) Span(S_1) is a subspace of Span(S_3).
$(1,2,3)$ and $(2,3,4)$ can both be written as linear combinations of the elements in S_3 (check). Any element in Span(S_1) is a linear combination of $(1,2,3)$ and $(2,3,4)$, and hence can also be written as a linear combination of the elements in S_3. **[4 marks]**

Question 7:

(a) See notes. **[5 marks]**

(b) In the notation of the notes:
$\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.
$\mathbf{w}_2 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \end{pmatrix}$. $\mathbf{v}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$.
$\mathbf{w}_3 = \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & 1 \end{pmatrix}$. $\mathbf{v}_3 = \frac{1}{\sqrt{21}}\begin{pmatrix} -2 & 2 \\ 2 & 3 \end{pmatrix}$.
$\mathbf{w}_4 = \begin{pmatrix} -\frac{1}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{2}{7} \end{pmatrix}$. $\mathbf{v}_4 = \frac{1}{\sqrt{7}}\begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}$.
Plus explanation of basis property. **[15 marks]**

Question 8:

(a)(i) Linear.

(ii) Not linear, e.g. $p(x) = 1$ and $q(x) = 2$.

(iii) Linear. **[6 marks]**

(b) Let the vectors be labelled $\mathbf{u}_i$ and $\mathbf{v}_j$ as in the notes, and $A = (a_{ij})$ be the desired matrix. Then
$f(\mathbf{u}_1) = -\mathbf{e}_1 + 3\mathbf{e}_2 - 2\mathbf{e}_3$
And we get that $a_{11} = -8$, $a_{21} = 5$, $a_{31} = 2$. Similarly
$f(\mathbf{u}_2) = 3\mathbf{e}_1 + 13\mathbf{e}_2 + 17\mathbf{e}_3$
And we get that $a_{12} = 24$, $a_{22} = -4$, $a_{32} = -17$.
Therefore $A = \begin{pmatrix} -8 & 24 \\ 5 & -4 \\ 2 & -17 \end{pmatrix}$. **[12 marks]**

(c) f cannot be an isomorphism as $\mathbb{R}^2$ and $\mathbb{R}^3$ do not have the same dimension. **[2 marks]**