Comments on Linear Algebra 1

In the last class we covered the main examples from the first exercise sheet. There were however a couple of other examples for which it may be useful to have some worked solutions.

4. This is an important exercise, and a good example of how the axioms for a vector space are used in proofs. Unfortunately, on the sheet we misstated the Exercise 1.10 as the already proven Proposition 1.7. Here is the correct statement and a proof.

   Theorem: Let $\mathbf{v_1}, \ldots, \mathbf{v_n}$ be vectors in a vector space V over $\mathbb{F}$, and W be the set of linear combinations of these vectors. Then W is a subspace of V.

   Pf: (Before proceeding with the proof, we note that our definition of a linear combination of vectors uses implicitly (V1) to avoid an excess of brackets.)

   (1) We first need to check that $\mathbf{0} \in W$.
   Now

   $$\begin{aligned} \mathbf{0} &= \textstyle\sum_{i=1}^{n} \mathbf{0} && \text{by (V3) (applied repeatedly)} \\ &= \textstyle\sum_{i=1}^{n} 0.\mathbf{v_i} && \text{by Theorem 1.5(3)} \end{aligned}$$

   and this is a linear combination of the $\mathbf{v_i}$ as required.

   (2) We next need to show that if $\mathbf{u}$ and $\mathbf{w} \in W$ then so is $\mathbf{u} + \mathbf{w}$.
   Let $\mathbf{u} = \sum_{i=1}^{n} \lambda_i \mathbf{v_i}$ and $\mathbf{w} = \sum_{i=1}^{n} \mu_i \mathbf{v_i}$ with $\lambda_i, \mu_i \in \mathbb{F}$. Then we have

   $$\begin{aligned} \mathbf{u} + \mathbf{w} &= (\textstyle\sum_{i=1}^{n} \lambda_i \mathbf{v_i} + \sum_{i=1}^{n} \mu_i \mathbf{v_i}) \\ &= \textstyle\sum (\lambda_i + \mu_i) \mathbf{v_i} && \text{by (V2) and (V6)} \end{aligned}$$

   which is an element of W, as required.

   (3) Finally, we must show that if $\mathbf{u} \in W$ and $\lambda \in \mathbb{F}$ then $\lambda \mathbf{u} \in W$.
   With $\mathbf{u}$ as above we have

   $$\begin{aligned} \lambda \mathbf{u} &= \lambda (\textstyle\sum_{i=1}^{n} \lambda_i \mathbf{v_i}) \\ &= \textstyle\sum_{i=1}^{n} \lambda (\lambda_i \mathbf{v_i}) && \text{by (V5)} \\ &= \textstyle\sum_{i=1}^{n} (\lambda \lambda_i) \mathbf{v_i} && \text{by (V7)} \end{aligned}$$

   which is an element of W, as required. □

8. As a further example of how the axioms for a vector space can be used, we complete the proof of Theorem 1.5, parts (2) and (4).

   Theorem: Given $\mathbf{v} \in V$, the element $-\mathbf{v}$ is unique, and we have $(-1).\mathbf{v} = -\mathbf{v}$.

   Pf: Suppose that $\mathbf{u}$ and $\mathbf{w}$ are two elements of V such that $\mathbf{v} + \mathbf{u} = \mathbf{0}$ and $\mathbf{v} + \mathbf{w} = \mathbf{0}$.
   Then

   $$\begin{aligned} \mathbf{u} &= \mathbf{u} + \mathbf{0} && \text{by (V3)} \\ &= \mathbf{u} + (\mathbf{v} + \mathbf{w}) && \text{by assumption} \\ &= (\mathbf{u} + \mathbf{v}) + \mathbf{w} && \text{by (V1)} \\ &= (\mathbf{v} + \mathbf{u}) + \mathbf{w} && \text{by (V2)} \\ &= \mathbf{0} + \mathbf{w} && \text{by assumption} \\ &= \mathbf{w} + \mathbf{0} && \text{by (V2)} \\ &= \mathbf{w} && \text{by (V3)} \qquad \text{as required.} \end{aligned}$$

To show that $(-1).\mathbf{v} = -\mathbf{v}$ we begin by considering

$$\begin{aligned} \mathbf{0} &= 0.\mathbf{v} && \text{by Theorem 1.5(3)} \\ &= (1 + (-1)).\mathbf{v} \\ &= 1.\mathbf{v} + (-1).\mathbf{v} && \text{by (V6)} \\ &= \mathbf{v} + (-1).\mathbf{v} && \text{by (V8).} \end{aligned}$$

Adding $(-\mathbf{v})$ to both side of the last equation we have

$$\begin{aligned} \mathbf{0} + (-\mathbf{v}) &= (\mathbf{v} + (-1).\mathbf{v}) + (-\mathbf{v}) \\ \mathbf{0} + (-\mathbf{v}) &= ((-1).\mathbf{v} + \mathbf{v}) + (-\mathbf{v}) && \text{by (V2)} \\ (-\mathbf{v}) + \mathbf{0} &= (-1).\mathbf{v} + (\mathbf{v} + (-\mathbf{v})) && \text{by (V2) and (V1)} \\ (-\mathbf{v}) &= ((-1).\mathbf{v} + \mathbf{0} && \text{by (V3) and (V4)} \\ (-\mathbf{v}) &= (-1).\mathbf{v} && \text{by (V3)} \qquad \text{as required.} \end{aligned}$$

□