Comments on Linear Algebra 2

In class we covered the main examples from the second exercise sheet. We include here answers for the remaining questions.

7. This question is a little bit more tricky than the ones we considered in class, as we are dealing with abstract vectors rather than numbers. Also, we have to find some way of using the fact that our original set of vectors is a basis.

   We wish to show that $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\}$ is a basis for a vector space V, given that the original set of three vectors was a basis. To do this we must show that the new vectors are linearly independent and span our space.

   First we show linear independence. Suppose that there exist $\lambda_i \in \mathbb{F}$ such that

   $$\lambda_1\mathbf{v}_1 + \lambda_2(\mathbf{v}_1 + \mathbf{v}_2) + \lambda_3(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = \mathbf{0}.$$

   Rearranging we see that

   $$(\lambda_1 + \lambda_2 + \lambda_3)\mathbf{v}_1 + (\lambda_1 + \lambda_2)\mathbf{v}_2 + \lambda_3\mathbf{v}_3 = \mathbf{0}.$$

   As our original set of vectors was a basis this implies that

   $$\begin{array}{cccl} \lambda_1 & +\lambda_2 & +\lambda_3 & = 0 \\ & \lambda_2 & +\lambda_3 & = 0 \\ & & \lambda_3 & = 0 \end{array}$$

   and hence that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Therefore our new set of vectors is linearly independent.

   For spanning we can argue in a similar fashion. Alternatively, we can note that V is 3-dimensional, as $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are a basis. Hence any 3 linearly independent vectors in V must form a basis, which shows that our new set is a basis of V.

8. Let V be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$. Suppose that V is finite dimensional, of dimension m say. The set $P_m(x)$ is a subspace of V, and has dimension $m + 1$. Hence there exists a basis of $m + 1$ polynomials for $P_m(x)$, which must be linearly independent in $P_m(x)$, and so also in V. However as V has dimension m, any set of $m + 1$ elements of V must be linearly dependent, which gives a contradiction. Therefore V must be infinite dimensional.

9. **Theorem:** Any linearly independent set $\mathbf{v}_1, \ldots, \mathbf{v}_r$ in a vector space V of dimension $n \geq r$ can be extended to a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of V.

   Pf: If $n = r$ we are done by Corollary 1.32. If $n > r$ then our original set of vectors do not span V, and hence we can choose $\mathbf{v}_{r+1} \in V \backslash \text{Span}\ (\{\mathbf{v}_1, \ldots, \mathbf{v}_r\})$. By Lemma 1.30 the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_{r+1}\}$ is linearly independent. Repeat this process until no more vectors can be chosen (this procedure must terminate as no set of more than n vectors can be linearly independent). At this final stage we have a set of linearly independent vectors which also must span V. Hence we have a basis of V as required. $\square$