Comments on Linear Algebra 3

In the examples class we went through the first three parts of question 3 on the third example sheet. This hand-out contains solutions for the remaining parts, some comments on the question, and the solution to question 4. Note that the detailed checks of linear independence and spanning are omitted — when doing questions you should include them!

Question 3

Recall that for each of the three remaining maps we are required to verify the rank-nullity formula. That is, if $f : U \to V$ is linear, then

$$\dim U = \dim \operatorname{im} f + \dim \ker f.$$

- The first map is

$$f : P_1 \to P_2 \qquad a_0 + a_1 x \longmapsto a_0 + a_1(x+1)^2.$$

In the class we saw that f was injective, and hence we know that $\dim \ker f = 0$. As $\dim P_1 = 2$ we have to verify that $\dim \operatorname{im} f = 2$. What is the general form of an element $f(p)$? If $p = a_0 + a_1 x$ then $f(p) = a_0 + a_1 + 2a_1 x + a_1 x^2$, which is of the form $a + 2bx + bx^2$ with $a, b \in \mathbb{R}$. Indeed, it is easy to see that given any a and b, we can find a polynomial p such that $f(p) = a + 2bx + bx^2$ (take $a_1 = b$ and $a_0 = a - b$). So we have to find a basis for the vector space of vectors of the form $a + 2bx + bx^2$.

I claim that such a basis is given by the polynomials 1 and $2x + x^2$. But this is easy, as they clearly span the desired space, and are also linearly independent. (Ensure that you are happy with the last sentence!) So $\dim \operatorname{im} f = 2$ as required.

- The next map is

$$f : \mathbb{R}^3 \to \mathbb{R}^3 \qquad (x, y, z) \longmapsto (x - y, z, z).$$

This is neither injective nor surjective, so we need to find bases for both $\operatorname{im} f$ and $\ker f$. If $f(x, y, z) = 0$ then we must have $z = 0$ and $x = y$, and conversely any such vector is in the kernel. So the kernel consists of vectors of the form $(x, x, 0)$. All such vectors are a scalar multiple of the vector $(1, 1, 0)$, and so this is a basis for $\ker f$. So we know that $\dim \mathbb{R}^3 = 3$ and $\dim \ker f = 1$. Thus to verify that the rank-nullity formula holds we must check that $\dim \operatorname{im} f = 2$.

Any vector in $\operatorname{im} f$ is of the form (a, b, b), and any such vector is in the image (e.g. equals $f(a, 0, b)$). So we need a basis of this space. It is easy to see that $(1, 0, 0)$ and $(0, 1, 1)$ gives such a basis, and hence $\dim \operatorname{im} f = 2$ as required.

- The final map to consider is

$$f : P_n \to P_n \qquad p(x) \longmapsto p(x) - p(1).$$

Again, we saw in the class that this is neither injective nor surjective. Let us first consider the kernel of f. It will be convenient to write out what the image of any given polynomial is under f. If $p = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ then

$$f(p) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x - (a_n + a_{n-1} + \cdots + a_1). \qquad (1)$$

This is zero if and only if $a_n = a_{n-1} = \cdots = a_1 = 0$; i.e. in the case $f = a_0$, a constant function. Any such function is just a scalar multiple of the function 1, and so this function is a basis for the kernel of f.

Now we know that $\dim P_n = n + 1$ and $\dim \ker f = 1$, so it remains to show that $\dim \operatorname{im} f = n$ in order to verify the rank-nullity formula. Thus we need to find a basis for polynomials of the form given in (1). I claim that such a basis is given by the n elements $x - 1$, $x^2 - 1$, …, $x^n - 1$. Clearly each of these elements lies in the image of f, and they are linearly independent. It is also routine to check that they span, and hence form a basis. This completely the verification of the rank-nullity formula in this case.

Remarks

The examples considered above are intended to illustrate the utility of the rank-nullity formula. Given f a linear map from U to V, it is usually easy to determine the dimension of U. Also, one often knows the dimension of either the image or the kernel of f (for example, if f is injective or surjective). However, it can sometimes be quite tricky to construct a basis for the remaining space (i.e. the kernel or the image of f).

Using the rank-nullity formula we can immediately determine the dimension of one of these spaces given the dimension of the other (and the dimension of U). This then makes it easier to find a basis for either space (if desired) as we just have to find the appropriate number of linearly independent vectors (by Corollary 1.32).

Question 4

To show that the composite is linear we need to check the two parts of the definition:

(i) For all $\mathbf{u}, \mathbf{v} \in U$ we have $f \circ g(\mathbf{u} + \mathbf{v}) = f \circ g(\mathbf{u}) + f \circ g(\mathbf{v})$.
(ii) For all $\mathbf{u} \in U$ and $\lambda \in \mathbb{F}$ we have $f \circ g(\lambda \mathbf{u}) = \lambda f \circ g(\mathbf{u})$.

We first consider (i). We have

$$\begin{aligned} f \circ g(\mathbf{u} + \mathbf{v}) &= f(g(\mathbf{u} + \mathbf{v})) \\ &= f(g(\mathbf{u}) + g(\mathbf{v})) \\ &= f(g(\mathbf{u})) + f(g(\mathbf{v})) = f \circ g(\mathbf{u}) + f \circ g(\mathbf{v}) \end{aligned}$$

by the linearity of f and g.

We next consider (ii). We have

$$\begin{aligned} f \circ g(\lambda \mathbf{u}) &= f(g(\lambda \mathbf{u})) \\ &= f(\lambda g(\mathbf{u})) \\ &= \lambda f(g(\mathbf{u})) = \lambda f \circ g(\mathbf{u}) \end{aligned}$$

again by the linearity of f and g.