Comments on Linear Algebra 5

In the examples class, we considered the matrices (a)–(c) on Example Sheet 5. The remaining examples are similar, but involve eigenspaces which are no longer simply one-dimensional. We shall consider each in turn.

The methods we use will be similar to those in the examples class, but the matrices have been chosen to make the calculations a bit simpler (in particular, we will not need to use Gaussian elimination to solve the resulting systems of equations). You should ensure that you understand the various basic methods from last year's algebra course (taking determinants and inverses of matrices, and the use of Gaussian elimination to solve equations) and are happy using them...

(d) We begin by finding the eigenvalues, and bases for the corresponding eigenspaces, of the matrix $A = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Step 1: Finding the eigenvalues.
We need to find the solutions of the equation $\det(A - \lambda I) = 0$.

$$\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 0 & -2 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix} = (2-\lambda)(3-\lambda)(3-\lambda)$$

and so the eigenvalues of A are $\lambda = 2$ and $\lambda = 3$.

Step 2: Finding the eigenvectors, and bases for the eigenspaces.
We need to find, for each eigenvalue λ, all solutions to the equation $A\mathbf{x} = \lambda\mathbf{x}$.
First consider $\lambda = 2$. Now $A\mathbf{x} = 2\mathbf{x}$ is simply

$$\begin{pmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and so we have to solve the system of equations

$$\begin{aligned} 2x_1 + 0x_2 - 2x_3 &= 2x_1 \\ 0x_1 + 3x_2 + 0x_3 &= 2x_2 \\ 0x_1 + 0x_2 + 3x_3 &= 2x_3. \end{aligned}$$

Clearly, the second and third equations imply that $x_2 = x_3 = 0$, and now the first equation implies that x_1 can be chosen freely. Thus the general solution of this system of equations is of the form $\mathbf{x} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$ where $x \in \mathbb{R}$. These are all scalar multiples of $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, which thus gives a basis of the eigenspace $S_A(2)$.

Next consider $\lambda = 3$. Now $A\mathbf{x} = 3\mathbf{x}$ is simply

$$\begin{pmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 3 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and so we have to solve the system of equations

$$\begin{aligned} 2x_1 + 0x_2 - 2x_3 &= 3x_1 \\ 0x_1 + 3x_2 + 0x_3 &= 3x_2 \\ 0x_1 + 0x_2 + 3x_3 &= 3x_3. \end{aligned}$$

Clearly, the second and third equations give us no conditions on any of the x_i, while the first equation says that $x_1 = -2x_3$. Thus the general solution of this system of equations is of the form $\mathbf{x} = \begin{pmatrix} -2z \\ y \\ z \end{pmatrix}$ where $y, z \in \mathbb{R}$. We need to find a basis for this set of eigenvectors.

It is no longer true that every such vector is a scalar multiple of one single vector, so our basis will have to have at least two elements in it. Given that we can choose y and z freely, it is natural to try to find simple eigenvectors with zeros in some of the entries, so let us consider the vectors $\mathbf{b} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. I claim that these form a basis for $S_A(3)$.

First that they span: Any eigenvector $\mathbf{x}$ of the form above can be written as $\mathbf{x} = z\mathbf{b} + y\mathbf{c}$, and so the vectors $\mathbf{b}$ and $\mathbf{c}$ span the eigenspace.
Second, that they are linearly independent: If $z\mathbf{b} + y\mathbf{c} = \mathbf{0}$ for some $y, z \in \mathbb{R}$, then by looking at the first coordinate in the sum (which is just $-2z$) we see that $z = 0$, and by looking at the second coordinate we see that $y = 0$. (This is why we chose basis vectors with lots of zeros, so that the calculations become easy!) Thus the vectors $\mathbf{b}$ and $\mathbf{c}$ are linearly independent.
As they are linearly independent and also span $S_A(3)$, they must form a basis of $S_A(3)$ as required.

Step 3: Applying the diagonalisation theorem.
We have now found bases for the one-dimensional space $S_A(2)$ and the two-dimensional space $S_A(3)$. As eigenvectors from distinct eigenspaces are linearly independent, we thus have three linearly independent eigenvectors for A. This is the same number as the number of rows (or columns) of A, and so we can apply the diagonalisation theorem.

Thus we can write down a matrix P such that $D = P^{-1}AP$ is diagonal with entries the eigenvalues of A. In particular, P is just the matrix $(\mathbf{abc})$ obtained from the three basis vectors; $P = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is the diagonal matrix of eigenvalues *written in the same order as the corresponding eigenvectors* (i.e., the eigenvalue for $\mathbf{a}$ first, then that for $\mathbf{b}$ and finally that for $\mathbf{c}$).

It is left to the reader to check that $P^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and that $D = P^{-1}AP$ as required.

(It is not *necessary* to do this if asked to find P, but is a useful check to ensure that you have not made a mistake.)

Step 4: Calculating A^{10}.
This is now easy:

$$A^{10} = (PDP^{-1})^{10} = PD^{10}P^{-1} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2^{10} & 0 & 0 \\ 0 & 3^{10} & 0 \\ 0 & 0 & 3^{10} \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

which can easily be calculated (and is left to the reader).

(e) We begin by finding the eigenvalues, and bases for the corresponding eigenspaces, of the matrix $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix}$.

Step 1: Finding the eigenvalues.
We need to find the solutions of the equation $\det(A - \lambda I) = 0$.

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 2 & 0 & 1-\lambda \end{pmatrix} = \lambda^2(1-\lambda)$$

and so the eigenvalues of A are $\lambda = 1$ and $\lambda = 0$.

Step 2: Finding the eigenvectors, and bases for the eigenspaces.
We need to find, for each eigenvalue λ, all solutions to the equation $A\mathbf{x} = \lambda\mathbf{x}$.
First consider $\lambda = 1$. Now $A\mathbf{x} = \mathbf{x}$ is simply

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and so we have to solve the system of equations

$$\begin{aligned} 0x_1 + 0x_2 + 0x_3 &= x_1 \\ 0x_1 + 0x_2 + 0x_3 &= x_2 \\ 2x_1 + 0x_2 + 1x_3 &= x_3. \end{aligned}$$

Clearly, the first and second equations imply that $x_1 = x_2 = 0$, and now the third equation implies that x_3 can be chosen freely. Thus the general solution of this system of equations is of the form $\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}$ where $x \in \mathbb{R}$. These are all scalar multiples of $\mathbf{a} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, which thus gives a basis of the eigenspace $S_A(1)$.
Next consider $\lambda = 0$. Now $A\mathbf{x} = 0\mathbf{x}$ is simply

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and so we have to solve the single non-zero equation

$$2x_1 + 0x_2 + x_3 = 0.$$

That is, the only constraint is that $x_3 = -2x_1$. Thus the general solution is of the form $\mathbf{x} = \begin{pmatrix} x \\ y \\ -2x \end{pmatrix}$ where $x, y \in \mathbb{R}$. We need to find a basis for this set of eigenvectors.

Again, not every such eigenvector is a scalar multiple of one single vector, so our basis will have to have at least two elements in it. Given that we can choose x and y freely, we will again choose simple eigenvectors with zeros in some of the entries. So let us consider the vectors $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$. I claim that these form a basis for $S_A(0)$.

First that they span: Any eigenvector $\mathbf{x}$ of the form above can be written as $\mathbf{x} = y\mathbf{b} + x\mathbf{c}$, and so the vectors $\mathbf{b}$ and $\mathbf{c}$ span the eigenspace.
Second, that they are linearly independent: If $y\mathbf{b} + x\mathbf{c} = \mathbf{0}$ for some $x, y \in \mathbb{R}$, then by looking at the first coordinate in the sum we see that $x = 0$, and by looking at the second coordinate we see that $y = 0$. Thus the vectors $\mathbf{b}$ and $\mathbf{c}$ are linearly independent.
As they are linearly independent and also span $S_A(0)$, they must form a basis of $S_A(0)$ as required.

Step 3: Applying the diagonalisation theorem.
We have now found bases for the one-dimensional space $S_A(1)$ and the two-dimensional space $S_A(0)$. As eigenvectors from distinct eigenspaces are linearly independent, we thus have three linearly independent eigenvectors for A. This is the same number as the number of rows (or columns) of A, and so we can apply the diagonalisation theorem.

Thus we can write down a matrix P such that $D = P^{-1}AP$ is diagonal with entries the eigenvalues of A. In particular, P is just the matrix $(\mathbf{abc})$ obtained from the three basis vectors; $P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is the diagonal matrix of eigenvalues *written in the same order as the corresponding eigenvectors* (i.e., the eigenvalue for $\mathbf{a}$ first, then that for $\mathbf{b}$ and finally that for $\mathbf{c}$).
It is left to the reader to determine P^{-1} and check that $D = P^{-1}AP$ as required.

Step 4: Calculating A^{10}.
This is now easy, for the same reasons as in the previous example, and is left as an exercise for the reader.