Linear Algebra 1

1. Satisfy yourself that $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ and similarly that $\mathbb{C}^n$ is a vector space over $\mathbb{C}$. Is $\mathbb{C}^n$ a vector space over $\mathbb{R}$?

2. Complete Example 1.3 to show that the set of polynomials (with real coefficients) of degree at most n is a vector space over $\mathbb{R}$.

3. Complete Example 1.4, to show that the space of functions from a set X to $\mathbb{R}$ is a vector space over $\mathbb{R}$.

4. Complete Exercise 1.10 by showing that a subspace of a vector space over $\mathbb{F}$ is itself a vector space over $\mathbb{F}$.

5. Consider the set containing the single element $\mathbf{0}$. We define an addition and scalar multiplication on this set by $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $\lambda.\mathbf{0} = \mathbf{0}$ for all $\lambda \in \mathbb{F}$. Show that this makes the set $\{\mathbf{0}\}$ into a vector space (which we call the **zero vector space**).

6. Consider the set $M(2,2)$ of 2×2 matrices with entries in $\mathbb{F}$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. Define addition of matrices in the usual way, and for $\lambda \in \mathbb{F}$ let
$$\lambda \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} \\ \lambda a_{21} & \lambda a_{22} \end{pmatrix}.$$
Show that this makes $M(2,2)$ into a vector space over $\mathbb{F}$. (In a similar way we can show that $M(a,b)$, the set of $a \times b$ matrices, is a vector space over $\mathbb{F}$.)

7. Which of the following are subspaces? Either show that they are, or explain why they fail to be.

   (a) The set of elements (x, y) with $x \geq 0$ and $y \in \mathbb{R}$, inside $\mathbb{R}^2$.

   (b) The set of elements (x, x) with $x \in \mathbb{R}$, inside $\mathbb{R}^2$.

   (c) The set of polynomials, with real coefficients, of degree at most n such that $f(0) = 0$, inside P_n.

   (d) The set of polynomials, with real coefficients, of degree at most n such that $f(0) = 1$, inside P_n.

   (e) The set of polynomials, with real coefficients, of the form $a_0 + a_1x + a_2x^2$ such that $a_0 + a_1 + a_2 = 0$, inside P_2.

   (f) The set of real-valued matrices of the form
   $$\begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}$$
   with a and b in $\mathbb{R}$, inside $M(2,2)$.

8. (Optional) Complete the proof of Theorem 1.5.