Linear Algebra 2

1. Check that the set of vectors \( \{ \mathbf{e}_1 = (1, 0, 0), \mathbf{e}_2 = (0, 1, 0), \mathbf{f} = (1, 1, 1) \} \) in Example 1.16 is linearly independent in \( \mathbb{R}^3 \).

2. Check (Example 1.18) that the above set of vectors form a basis of \( \mathbb{R}^3 \), but that the set \( \{ \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{f} \} \) (where \( \mathbf{e}_3 = (0, 0, 1) \)) does not.

3. Verify that the standard bases given in Examples 1.18, 1.22 and 1.23, for \( \mathbb{R}^n \), \( P_n \) and \( M(m, n) \) respectively, are bases over \( \mathbb{R} \).

4. Let \( \mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{C}^n \) be the vector with a 1 in the \( i \)th entry and zeros elsewhere.
   (a) Show that \( \{ \mathbf{e}_1, \ldots, \mathbf{e}_n \} \) is a basis for \( \mathbb{C}^n \) as a vector space over \( \mathbb{C} \).
   (b) Show that \( \{ \mathbf{e}_1, \ldots, \mathbf{e}_n \} \) is not a basis for \( \mathbb{C}^n \) as a vector space over \( \mathbb{R} \).
   (c) Show that \( \{ \mathbf{e}_1, \ldots, \mathbf{e}_n, i\mathbf{e}_1, \ldots, i\mathbf{e}_n \} \) is a basis for \( \mathbb{C}^n \) as a vector space over \( \mathbb{R} \).

5. Which of the following sets are bases for \( \mathbb{R}^3 \)?
   (a) \( \{(1, 0, 0), (2, 2, 0), (3, 3, 3)\} \).
   (b) \( \{(2, 1, 3), (6, 3, 9), (3, 1, 2)\} \).
   (c) \( \{(3, 1, -4), (2, 5, 6), (5, 6, 2)\} \).
   (d) \( \{(1, 6, 4), (2, 4, -1), (-1, 2, 5)\} \).

6. Which of the following sets are bases for \( P_2 \)?
   (a) \( \{1 + x + x^2, x + x^2, x^2\} \).
   (b) \( \{1 - 3x + 2x^2, 1 + x + 4x^2, 1 - 7x\} \).
   (c) \( \{-1 + x + 2x^2, 1 - x + x^2, 3x^2\} \).
   (d) \( \{-4 + x + 3x^2, 6 + 5x + 2x^2, 8 + 4x + x^2\} \).

7. Let \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \) be a basis for a vector space \( V \). Show that \( \{ \mathbf{v}_1, \mathbf{v}_2 + \mathbf{v}_1, \mathbf{v}_3 + \mathbf{v}_2 + \mathbf{v}_1 \} \) is also a basis for \( V \).

8. (Optional) Show that the space of all functions \( f : \mathbb{R} \to \mathbb{R} \) is not a finite dimensional vector space over \( \mathbb{R} \). (Hint: use polynomials.)

9. (Optional) Complete the proof of Corollary 1.33.