Linear Algebra 3

1. For each of the following maps, either prove the map is linear or give an example to show that linearity fails.

   (a) $f : \mathbb{R}^2 \to \mathbb{R}^2 \qquad (x, y) \longmapsto (x + 2y, 2x - y)$.

   (b) $f : \mathbb{R}^2 \to \mathbb{R}^3 \qquad (x, y) \longmapsto (x + 2y, x, 2xy)$.

   (c) $f : \mathbb{R}^n \to \mathbb{R}^m \qquad \mathbf{x} \longmapsto A\mathbf{x}$ where A is an $m \times n$ matrix with entries in $\mathbb{R}$.

   (d) $f : P_n \to P_{n+1} \qquad p(x) \longmapsto xp(x)$.

   (e) $f : P_n \to P_n \qquad p(x) \longmapsto p(3x - 5)$.

   (f) $f : P_n \to \mathbb{R} \qquad p(x) \longmapsto p(1)$.

   (g) $f : P_n \to P_n \qquad p(x) \longmapsto 2p + \frac{d}{dx}(p)$.

   (h) $f : P_2 \to P_2 \qquad a_0 + a_1x + a_2x^2 \longmapsto (a_0 + 1) + (a_1 + 1)x + (a_2 + 1)x^2$.

   (i) $f : M(n, n) \to \mathbb{R} \qquad A \longmapsto \mathrm{tr}(A)$.

   (j) $f : M(m, n) \to M(n, m) \qquad A \longmapsto A^T$.

   (k) $f : M(n, n) \to \mathbb{R} \qquad A \longmapsto \det(A)$.

   (l) $f : M(2, 2) \to \mathbb{R} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto 3a - b + 2c + d$.

   (m) $f : M(2, 2) \to \mathbb{R} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto 3a^2 + d^2$.

2. For each of the following linear maps determine whether it is surjective, injective, both or neither, giving proofs or examples in each case.

   (a) $f : \mathbb{R}^3 \to \mathbb{R}^2 \qquad (x, y, z) \longmapsto (x, z)$.

   (b) $f : \mathbb{R}^3 \to \mathbb{R}^3 \qquad (x, y, z) \longmapsto (y, x, z)$.

   (c) $f : M(2, 2) \to \mathbb{R} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto a - d$.

   (d) $f : P_1 \to P_2 \qquad a_0 + a_1x \longmapsto a_0 + a_1(x + 1)^2$.

   (e) $f : \mathbb{R}^3 \to \mathbb{R}^3 \qquad (x, y, z) \longmapsto (x - y, z, z)$.

   (f) $f : P_n \to P_n \qquad p(x) \longmapsto p(x) - p(1)$.

3. For each of the maps in Question 2, verify that the rank-nullity formula holds.

4. (Optional) If $f : V \to W$ and $g : U \to V$ are linear maps, prove that the composition $f \circ g : U \to W$ given by $(f \circ g)(\mathbf{x}) = f(g(\mathbf{x}))$ is also linear.