Linear Algebra 6

1. Verify that the Cayley-Hamilton theorem holds for the matrix
\[
\begin{pmatrix}
-1 & 4 & -2 \\
-3 & 4 & 0 \\
-3 & 1 & 3
\end{pmatrix}.
\]

2. For each of the following functions, either prove that they give a real inner product, or give an example to show how they fail.

(a) \(\langle -,- \rangle\) on \(\mathbb{R}^3\) given by \(\langle x,y \rangle = 2x_1y_1 + 4x_2y_2 + 6x_3y_3\).

(b) \(\langle -,- \rangle\) on \(\mathbb{R}^3\) given by \(\langle x,y \rangle = 2x_1y_1 + 4x_2y_2 - 6x_3y_3\).

(c) \(\langle -,- \rangle\) on \(\mathbb{R}^3\) given by \(\langle x,y \rangle = x_1y_2 + x_2y_3 + x_3y_1\).

(d) \(\langle -,- \rangle\) on \(P_2\) given by \(\langle p,q \rangle = \int_{-1}^{1} p(x)q(x)dx\).

(e) \(\langle -,- \rangle\) on \(P_2\) given by \(\langle p,q \rangle = p_0q_0 + p_1q_1 + p_2q_2\), where \(p(x) = p_0 + p_1x + p_2x^2\) and \(q(x) = q_0 + q_1x + q_2x^2\).

(f) \(\langle -,- \rangle\) on \(P_2\) given by \(\langle p,q \rangle = p_0q_0 + p_1q_1\), where \(p(x) = p_0 + p_1x + p_2x^2\) and \(q(x) = q_0 + q_1x + q_2x^2\).

(g) \(\langle -,- \rangle\) on \(M(2,2)\) given by \(\langle A,B \rangle = \text{tr}(A + B)\).

3. Calculate the inner product \(\langle A,B \rangle\) of \(A = \begin{pmatrix} 1 & 3 \\ 5 & 6 \end{pmatrix}\) and \(B = \begin{pmatrix} 2 & -4 \\ 0 & -1 \end{pmatrix}\) using the inner product given on matrices in the lectures (Example 4.4).

4. Calculate the inner product \(\langle p,q \rangle\) of \(p = 1 + x^2\) and \(q = 2 - 3x\) for each of the functions 2(d)-(f) which is an inner product.