Linear Algebra 7

1. Verify that the Cauchy-Schwarz inequality holds for the pair of matrices

$$\begin{pmatrix} 1 & 7 & -3 \\ 4 & 4 & 2 \\ 3 & -1 & 6 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 3 \\ 1 & -1 & -1 \end{pmatrix}$$

with respect to the inner product $\langle A, B\rangle = \mathrm{tr}(B^T A)$.

2. Which of the following sets are orthogonal. Which are orthonormal?

   (a) $\{(1,2,2,1), (-2,3,-2,0), (1,0,-1,1)\} \subset \mathbb{R}^4$ with respect to the usual scalar (ie "dot") product.

   (b) $\{(1,0), (0,1)\} \subset \mathbb{R}^2$ with respect to the inner product $\langle \mathbf{u}, \mathbf{v}\rangle = 4u_1v_1 + 6u_2v_2$.

   (c) $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}, \begin{pmatrix} 0 & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{pmatrix} \right\} \subset M(2,2)$ with respect to the inner product $\langle A, B\rangle = \mathrm{tr}(B^T A)$.

3. Use Gram-Schmidt to make an orthonormal basis for P_2 with inner product $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)dx$ from the set of basis elements $\{1, x, x^2\}$. *The resulting elements are called the first three (normalised) Legendre poynomials.* Then write each of the following vectors as a linear combination of these new basis elements.

   (a) $-2 + 5x + x^2$.

   (b) $x^2 - 9$.

   (c) $x + 13$.

4. Use Gram-Schmidt to make an orthonormal basis for $\mathbb{R}^4$ with the usual scalar product from the set of basis elements

$$\{(0,2,1,0), (1,-1,0,0), (1,2,0,1), (1,0,0,1)\}.$$

Write the vector $(3,7,4,5)$ as a linear combination of these new basis elements.

5. For each A below, find orthogonal matrices P such that $P^T AP$ is diagonal.

   (a) $\begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix}$.

   (b) $\begin{pmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{pmatrix}$.

   (c) $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$.