Cantor's diagonal argument

All of the infinite sets we have seen so far have been 'the same size'; that is, we have been able to find a bijection from $\mathbb{N}$ into each set. It is natural to ask if *all* infinite sets have the same cardinality. Cantor showed that this was not the case in a very famous argument, known as *Cantor's diagonal argument*.

We have not yet seen a formal definition of the real numbers — indeed such a definition is rather complicated — but we do have an intuitive notion of the reals that will do for our purposes here. We will represent each real number by an infinite decimal expansion; for example

$$\begin{aligned} 1 &= 1.00000\ldots \\ \tfrac{1}{2} &= 0.50000\ldots \\ \tfrac{1}{3} &= 0.33333\ldots \\ \tfrac{\pi}{4} &= 0.78539\ldots \end{aligned}$$

The only way that two distinct infinite decimal expansions can be equal is if one ends in an infinite string of 0's, and the other ends in an infinite sequence of 9's. For example

$$1.00000\ldots = 0.99999\ldots$$

We shall chose to represent each such number by the "zeros" version. In this way we can represent each real number by a *unique* decimal expansion.

We are now in a position to prove

Theorem 1 (Cantor) *The set $I = \{x \in \mathbb{R} : 0 < x < 1\}$ does not have cardinality $\aleph_0$.*

Proof: We have to show that *there does not exist* a bijection $f : \mathbb{N} \to I$. It is not good enough to show that any particular map is not a bijection. This sounds difficult — there are very many maps from $\mathbb{N}$ to I!

Suppose that we do have a bijection $f : \mathbb{N} \to I$. We will show that this assumption leads to a contradiction (i.e. a logical impossibility), and hence deduce that it must be false. This will be enough to prove the theorem.

Given our assumed bijection f, we can now list the elements of I in the order that they are mapped to be f:

$$\begin{array}{ll} 1 \longmapsto & 0.a_{11}a_{12}a_{13}a_{14}\ldots \\ 2 \longmapsto & 0.a_{21}a_{22}a_{23}a_{24}\ldots \\ 3 \longmapsto & 0.a_{31}a_{32}a_{33}a_{34}\ldots \\ 4 \longmapsto & 0.a_{41}a_{42}a_{43}a_{44}\ldots \\ \vdots & \vdots \end{array} \tag{1}$$

where each $a_{ij} \in \{0, 1, \ldots, 9\}$. As f is a bijection, every number in I occurs somewhere in this list. So if we can construct an element of I that does not occur in the list, then we get a contradiction, as required.

So how do we construct such an element? Let b_1 be an element of $\{1, 2, \ldots, 8\}$ such that $b_1 \neq a_1$; b_2 be an element of $\{1, 2, \ldots, 8\}$ such that $b_2 \neq a_2$; b_3 be an element of $\{1, 2, \ldots, 8\}$

such that $b_3 \neq a_3$ and so on. Now consider the infinite decimal expansion $b = 0.b_1b_2b_3\ldots$. Clearly $0 < b < 1$, and b does not end in an infinite string of 9's. So b must occur somewhere in our list above (as it represents an element of I). Therefore there exists $n \in \mathbb{N}$ such that $n \longmapsto b$, and hence we must have

$$0.a_{n1}a_{n2}a_{n3}\ldots = 0.b_1b_2b_3\ldots$$

But $a_{nn} \neq b_n$ (by construction) and so we cannot have the above equality. This gives the desired contradiction, and thus proves the theorem.

□

Remarks

(i) The above proof is known as the diagonal argument because we constructed our element b by considering the diagonal elements in the array (1).

(ii) It is possible to construct ever larger infinite sets, and thus define a whole hierarchy of cardinalities. A natural question to ask is whether the cardinality c of the set I above is the next largest cardinality after $\aleph_0$? The belief that there are indeed no cardinalities between $\aleph_0$ and c is known as the *continuum hypothesis*.

At the start of the twentieth century, mathematicians and logicians tried to construct a list of axioms (i.e. basic assumptions) for set theory from which all other results could be deduced. The most famous and widely used of these are the eight Zermolo-Fraenkel (ZF) axioms. Remarkably, it is possible to prove that the truth of the continuum hypothesis *depends on our model of set theory* (that is, on the choice of axioms that we make at the beginning), and that the ordinary ZF axioms are not enough to distinguish between the two possibilities. Thus, in some sense, the truth of the continuum hypothesis depends on what kind of mathematical universe we choose to live in!