Mathematical communication 3

This is an assessed coursework, and will count towards your final grade. Solutions should be handed in to the general office (C123) by **3:00pm on Tuesday 4th March**. Late submissions will be penalised.

1. Use quantifiers and predicates to symbolise each of the following statements. (Remember to define what each predicate means, and to give the set for which it is defined.)

   (a) Every library book is either borrowed or reserved.

   (b) If all coursework marks are low then no student will be happy.

   [10]

2. Let $p(x)$ be the predicate "$x > 0$" and $n(x)$ be the predicate "$x < 0$" where $x \in \mathbb{Z}$. Determine whether each of the following is true or false.

   (a) $(\forall x)(p(x) \vee (\neg n(x)))$.

   (b) $(\exists x)(p(x) \wedge n(x))$.

   (c) $((\exists x)p(x)) \wedge ((\exists x)\neg n(x))$.

   (d) $(\exists x)(\exists y)(p(x) \wedge n(y))$.

   (e) $((\forall x)p(x)) \vee ((\exists x)n(x))$.

   [15]

3. Let $p(x)$ be the predicate "x is a prime number" and $q(x)$ be the predicate "x is an odd number" where $x \in \mathbb{N}$. By using the rules for negating predicates and propositions, write

   $$\neg((\forall x)(p(x) \rightarrow q(x)))$$

   in good English in two different ways.

   [10]

4. Let $p(x, t)$ be the predicate "you can fool x at time t" where x comes from the set of people and t comes from the set of moments in time. Symbolise the following:

   > If you can fool some of the people all of the time, and all of the people some of the time, it does not imply that you can fool all of the people all of the time.

   [10]

5. (a) In one of the lectures we proved that $\sqrt{2}$ is irrational. By modifying the proof of this, show that $\sqrt{3}$ is also irrational. (You may assume that if b is an integer with b^2 divisible by 3 then b is also divisible by 3.)

   (b) Why does a similar argument not prove that $2 = \sqrt{4}$ is irrational? (You should explain at what stage the modified proof would fail.)

   [20]

6. Consider the following almost complete proof:

   > We can write a in the form $a = 3k + 1$ or $a = 3k + 2$ where $k \in \mathbb{N}$.
   > First suppose that $a = 3k + 1$.
   > Then $a^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$ is of the form $a^2 = 3l + 1$ with $l \in \mathbb{N}$.
   > Now suppose that $a = 3k + 2$.
   > Then $a^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$ is of the form $a^2 = 3l + 1$ with $l \in \mathbb{N}$.
   > In either case we see that ...

   Which *two* of the following statements does this prove? For each such statement say whether the proof is direct, contrapositive, or by contradiction.

   (a) If a is divisible by 3 then a^2 is divisible by 3.

   (b) If a is not divisible by 3 then a^2 is not divisible by 3.

   (c) If a^2 is divisible by 3 then a is divisible by 3.

   (d) If a^2 is not divisible by 3 then a is not divisible by 3.

   (e) If a is not divisible by 3 then a^2 is divisible by 3.

   [10]

7. (a) Prove that if x is irrational then $\sqrt{x}$ is irrational by considering the contrapositive.

   (b) Is

   $$\frac{\sqrt{2}}{1 + \sqrt{2}}$$

   rational or irrational? Give reasons for your answer (you may assume that $\sqrt{2}$ is irrational).

   (c) Which two of the following are true? Identify which can proved directly, and which can be proved by contradiction. Give proofs or counterexamples in each case.

   i. If x and y are rational then $x + y$ is rational.

   ii. If x is rational and y is irrational then $x + y$ is irrational.

   iii. If x and y are irrational then $x + y$ is irrational.

   [25]