Mathematical communication 4

This is an assessed coursework, and will count towards your final grade. Solutions should be handed in to the general office (C123) by **3:00pm on Monday 31st March**. Late submissions will be penalised.

1. The *handshaking lemma* says that in any finite collection of handshakes, the total number of hands shaken (counting repeats) will be even. (This is true because every shake involves two hands.) We shall assume that no person can shake hands with themself, and each only shakes hands with another person at most once. *In each of the following parts give reasons for your answers.*

   (a) Suppose that there are $n > 1$ people at a party. If n is odd, is it possible for every person to shake an odd number of hands?

   (b) Suppose that there are 5 people at a party. Is it possible for every person to shake a different number of hands?

   (c) Suppose that there are $n > 1$ people at a party. Generalise your answer to part (b) to this case. (Hint: have you seen something similar to this?)

   (d) Suppose there are infinitely many people at a party. Is it possible for every person to shake an odd number of hands?

   (e) Suppose there are infinitely many people at a party. Show that it is possible for each person to shake a different number of hands.

   [30]

2. Consider the following "proof" that everybody has the same birthday.

   We will prove the result by induction. Let $P(n)$ be the statement "in any group of n people, everyone has the same birthday". We first note that $P(1)$ is obviously true, as in any group of one person, all people in that group share the same birthday.

   Now suppose that $P(k)$ is true for some $k \in \mathbb{N}$. We must show that this implies that $P(k+1)$ is true also. To prove that $P(k+1)$ is true we must show that in every group of $k+1$ people, everyone has the same birthday. Suppose we have a group of $k+1$ people. Let A be a set of k people from this group, and B be another set of k people from the group, with $A \neq B$. As $P(k)$ is true by assumption, we know that everyone in A has the same birthday. In the same way, we know that everyone in B has the same birthday.

   Now A contained all of the group except one person x, who must have appeared in B. Similarly, B contained all of the group except one person y, who must have appeared in A. So x shares a common birthday with everyone in $A \cap B$, and y shares a common birthday with everyone in $A \cap B$. Therefore everyone has the same birthday. We have shown that $P(k)$ true implies that $P(k+1)$ is true. Now the result follows by induction. $\square$

   Explain why this argument is not correct. **[10]**

3. Prove for all positive integers n that

$$\sum_{i=1}^{n} i^2 = \frac{1}{6} n(n+1)(2n+1).$$

   [15]

4. Prove by induction that every set of $n \geq 1$ elements has exactly 2^n distinct subsets (including the empty set).

   [15]

5. Let (a_n) be the sequence of numbers defined recursively by $a_1 = 0$, $a_2 = 1$, and

$$a_n = 3a_{n-1} - 2a_{n-2}$$

   for $n \geq 3$. Prove that $a_n = 2^{n-1} - 1$ for all $n \geq 3$. **[15]**

6. Choose $n \geq 3$ distinct points on a circle and connect them in order to form a closed polygon. Show that the angles inside the polygon sum to $(n-2)\pi$ radians. (You may assume standard facts about triangles.) **[15]**