An introduction to Mathematical Communication

The object of this course is to introduce you to some of the basic tools needed by any mathematician, and much of what we consider will apply to almost every future course that you will take.

We will also introduce a new style of mathematics, which is called Pure Mathematics at university. This is rather different from what is studied under the same name at school (which at university is known as Applied Mathematics). One of the key aims is to introduce you to the notion of rigorous proof: how to reason carefully from an initial set of hypotheses to a conclusion.

The need for care

At school, mathematics consists of calculations. As long as these are performed carefully, there is little that can go wrong. However, mathematics at university is often involved with determining new facts about how the mathematical world is constructed. Here it is a lot easier to be led astray.

1. Imagine you have in front of you two sealed envelopes, each containing a cheque. You are told that each cheque is made out for some positive real number of pounds, and that one is exactly twice the value of the other.
I hand you the first envelope and you open it and read the amount on the cheque. Now you must choose which envelope to keep. Should you stick with the initial envelope or switch?

Here is a possible argument:
You see a cheque for x pounds. It is equally likely that the other cheque is for $x/2$ or $2x$ pounds. Thus if you switch, you expect to have $(1/2)(x/2) + (1/2)(2x) = 5x/4$ pounds. Hence you should switch envelopes. (If you are unfamiliar with calculating expectations, consider that there is a 50% chance of losing $x/2$ pounds and a 50% chance of gaining x pounds. Therefore you should switch.)

However, this argument applies regardless of which envelope you started with. So it is *always* better to switch!

2. Now imagine we want to sum the infinite series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

It turns out that this can be done. But for other infinite series such as

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

the sum does not make sense. How can we determine which series are possible to sum up?

Clearly there must be (and is) something wrong with the first argument, though it is hard to see precisely what. One of the skills a mathematician must learn is how to reason carefully, and avoid making unconscious assumptions.

In the second case we come across an example of a mathematical problem which goes beyond our intuition. It is easy to come up with nonsensical answers unless we are very careful in our analysis. Mathematics has developed powerful techniques for dealing with situations such as sums of infinite series, but they need to be used with caution.

These two examples suggest that we need to be careful in what we do. Misapplication of mathematical techniques can give us the wrong answer. But another problem that we can face is that the right answer is surprising or contrary to our common sense:

1. Suppose we are given 4 dice A, B, C, and D. Two players each pick a die and roll it, and the highest number wins. When A is compared with B, it wins two-thirds of the time. When B is compared with C it wins two-thirds of the time. When C is compared with D it wins two-thirds of the time. When A is compared to D, which is most likely to win?

2. Now imagine you are given a sphere. After cutting it into a finite number of pieces and rearranging them, can you obtain two spheres each the same size as the original one?

For the first question, you probably expect that A is most likely to win. But if we consider the following four dice:

$$A : 4, 4, 4, 4, 0, 0 \quad B : 3, 3, 3, 3, 3, 3 \quad C : 2, 2, 2, 2, 6, 6 \quad D : 5, 5, 5, 1, 1, 1$$

then it is easy to verify that they satisfy the conditions and yet D beats A two-thirds of the time.

It seems that the answer to the second question is “no”. Indeed, with any normal definition of “cutting into pieces” it is obviously impossible to double the sphere. However, in a branch of mathematics called analysis, there is a theory of ‘cutting’ which does allow the two spheres to be made — though of course this can never be carried out in the real world!

These two examples illustrate that a mathematician must learn to be wary of following the ‘intuitively obvious’. Pure mathematics is concerned only with what follows from the initial hypotheses, not what is ‘expected’ to be true. Our ‘common sense’ can lead us astray.

In practice, most of what you will see during your degree will be in line with your common sense; in applied maths in particular we desire results that correspond with reality. Thus examples as surprising as those above are unlikely to arise. However, it is important to learn to take a certain amount of care. This course is mainly devoted to the ‘grammar of mathematics’; that is, the language and methods with which best to express precisely what you wish to say.

Unlike in school mathematics, you will find that many results are proved using lengthy arguments in English as well as symbols. For this reason we shall initially have to spend a little time considering how to use English to express mathematics.

English in mathematics

When writing mathematics, your aim is to convey your arguments in a clear, coherent and precise manner — be it to your examiner, another mathematician, or even yourself! There are certain aspects of mathematical writing that differ from ordinary English, and we shall consider some of these shortly. But we begin with some general remarks that apply to almost any written work.

Keep it simple.

The object of technical writing is to convey ideas, not to demonstrate how large your vocabulary is, or your mastery of complex sentence structure. A common error is to write one long overly complicated sentence where several simpler ones would suffice. Such writing can confuse the reader (or, worse, the writer), and should be avoided.

Do not be too informal.

Different styles of language are appropriate for different occasions. Your mathematics should be more formal than your everyday speech or writing. This is not just for the sake of form — you will often be expressing complicated ideas, and the extra care needed to express them properly will often help to clarify matters, both for the author and the reader. A classic (if slightly old-fashioned) guide to writing plain English is "The Elements of Style" by William Strunk. This can be found in the library, or on the web at `http://www.bartleby.com/141/`.

We now turn to the particular problems that arise when writing mathematics. Consider the problem of giving a precise definition of what it means for a function on the real numbers to be continuous at a point. Intuitively, this should mean something like "possible to draw without removing the pen from the page", but this is not precise enough a definition to be able to work with. Here are three attempts to be more precise.

1. A function f from the real numbers to the reals is *continuous at a point a* if, for all real numbers greater than zero, there is another real number greater than zero so that the modulus of the difference between the values of the function at a and any other point is less than the first number, whenever the modulus of the difference between a and the other point is less than the second number.

2. $f : \mathbb{R} \to \mathbb{R}$ *continuous at* $a \in \mathbb{R} \Leftrightarrow$

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x)((|x - a| < \delta) \longrightarrow (|f(x) - f(a)| < \epsilon)).$$

3. A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous at* $a \in \mathbb{R}$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$.

Although all three examples express the same idea, the first two are both rather hard to read, while the third is much easier to follow. (Do not worry if you do not understand

any of the definitions; this is not a course in analysis...). Also, any attempt to express such a definition entirely in words is likely to introduce imprecision and ambiguity. Examples such as the second one do arise in mathematics (and we shall see similar statements later in the course), but it is much more common to express oneself in a mixture of words and symbols. A common mistake among students is to try to present their work using as little ink as possible. Without the use of a little English (in proper sentences!) you run the risk of producing little more than impenetrable hieroglyphics.

Mathematics differs from other subjects in that much of its technical vocabulary is made up of standard English words. However, these often have a quite different meaning from that usually intended in everyday speech. Most of the time it will be obvious that the specialist meaning is to be understood (and these new meanings will be introduced as they are needed), but there are a number of common words that can cause confusion. We shall briefly consider some of the more common of these.

Or has two distinct meanings in ordinary English. An example of the first (*exclusive or*) meaning is given by "I will pay you £15 or return my ticket before the show". Here either £15 will be paid or the ticket returned, but not both. However, in "most children learn French or German at school", the possibility that some children learn both languages is not excluded. This is an example of the *inclusive or*. In mathematics this latter, inclusive, meaning is always intended, unless specifically stated otherwise.

Some also differs slightly from its common meaning. In standard English "some people are never satisfied" would usually be taken to mean that several people, but not all, are never satisfied. However, a mathematician will also allow the two further possibilities that precisely one person is never satisfied, or that every person is never satisfied. As long as there is at least one example of a y having property x, then "some y are x" will be a true statement.

Any has two different meanings which can make its use in mathematics somewhat ambiguous. Consider the phrase "Show that any quadratic equation is the product of two linear factors". What is intended here is that you should show that *every* such equation has two factors, but another possible reading is that a single example of a quadratic equation should be chosen, and then shown to have two factors. Usually it will be clear whether you are intended to consider an arbitrary example or find a particular instance which satisfies the requirements, but in cases like the above it may sometimes be better to substitute the word "every" to avoid confusion.

We next consider a pair or words which are by far the most commonly confused terms among students, despite both having precisely the same meaning as in standard English. As they are both intimately related to the concept of proof, we shall consider each of them in detail.

A **necessary** condition is one that *must* hold for the stated conclusion to be true. It may not be enough in itself to ensure the desired consequence, but it certainly cannot be omitted. For example, to become President of the United States, it is *necessary* to be a citizen of the US. However, most citizens are not (and never will be) President.

A **sufficient** condition is one that *guarantees* that the stated conclusion holds. Often it will be possible to achieve the conclusion without meeting this condition, but it will always be enough just to satisfy it. For example, to become President of the United States it is *sufficient* to be the only eligible candidate to gain any votes in a presidential election. Of course, most Presidents are elected without meeting this condition.

Confusion of necessary and sufficient conditions is remarkably widespread. It is a very good exercise to watch almost any debate (for example, between politicians on television), and spot the occasions on which the two notions are confused. In mathematics it is common to be asked to prove that a condition is necessary, or sufficient, or both, and it is important to learn to distinguish between the various possibilities.

To complicate matters, there are another set of terms that can be used to describe such conditions. Instead of "x is a sufficient condition for y to hold" it is common to write "if x then y". Similarly, instead of "x is a necessary condition for y to hold" it is possible to write "y only if x". The latter possibility is most commonly seen in the phrase "y if and only if x"; that is "x is a necessary *and* sufficient condition for y". Sometimes "if and only if" may be abbreviated "iff", but this is not to be recommended as it can easily be misread and cause confusion.

As an example, consider the inequality

$$x^2 + 2x \geq 0. \tag{1}$$

If $x \geq 0$ then inequality (1) holds. Alternatively, we could say that the inequality holds only if $x \neq -1$. A complete description of the set of elements x satisfying (1) could be given by saying that (1) holds if and only if $x \leq -2$ or $x \geq 0$.

Common mathematical terminology

One of the problems facing any new mathematics student is the large array of standard notation appearing in their courses and books. It is often assumed that this material is already familiar, and so much of it is rarely explained. The following is intended to give an introduction to the basic symbols and terminology that are likely to arise in almost any course.

We begin with a description of the different types of results that you may come across. Most familiar of these are **theorems**, which are usually the most important results in the subject. Less important results are often called **propositions**. In the course of proving a theorem or proposition, the proof may be split into several intermediate steps (which are sometimes of interest in their own right). Such preparatory results are called **lemmas**. Following a theorem or proposition there may be a number of **corollaries**, which are consequences of the earlier result. Finally, statements that are believed to be true — but for which a proof cannot (yet) be found — are called **conjectures**.

Mathematicians are fond of using a wide variety of symbols, many of which will be unfamiliar to you. Although the majority of these will be explained as they arise, there are certain basic symbols which will in general be used without comment. The most common

of these are the letters of the Greek alphabet, which are given in Table 1. These usually appear in their lower case form (given in the left-hand columns below). Several of these have variants (also listed) which can sometimes appear in books and articles.

Table 1: The letters of the Greek alphabet

α	A	alpha	ι	I	iota	ρ, ϱ	P	rho
β	B	beta	κ	K	kappa	σ, ς	Σ	sigma
γ	Γ	gamma	λ	Λ	lambda	τ	T	tau
δ	Δ	delta	μ	M	mu	υ	Υ	upsilon
ϵ, ε	E	epsilon	ν	N	nu	ϕ, φ	Φ	phi
ζ	Z	zeta	ξ	Ξ	xi	χ	X	chi
η	H	eta	o	O	omicron	ψ	Ψ	psi
θ, ϑ	Θ	theta	π, ϖ	Π	pi	ω	Ω	omega

There are a number of other commonly occurring symbols that you may come across, and we list a few of these in Table 2. Some of these will be explained in greater detail later in this course.

Table 2: Other commonly occurring symbols

$\Rightarrow$	Implies that ...	$\mathbb{Z}$	The integers.
$\Leftarrow$	Is implied by ...	$\mathbb{N}$	The natural numbers.
$\Leftrightarrow$	Is equivalent to ...	$\mathbb{Q}$	The rational numbers.
$\therefore$	Therefore ...	$\mathbb{R}$	The real numbers.
$\forall$	For all ...	$\mathbb{C}$	The complex numbers.
$\exists$	There exists ...	$\square$	Used to denote the end of a proof.

Finally it is important to remember that equations, just like sentences, have certain 'rules of grammar' which must be followed. Most importantly, the order and position in which symbols are written down is usually crucial. The expression $\sin\theta$ is fine, while $\theta \sin$ is nonsense. In practice, a more common error is to write down an expression that is ambiguous; for example, does $6 \ / \ 1 - 5 \ / \ 7$ equal $\frac{1}{7}$, or $5\frac{2}{7}$, or 21, or ...?

In order to avoid ambiguity, it is important to learn to use brackets appropriately. These come in various forms:

$$(\ , \ ), \ [\ , \ ], \ \{ \ , \ \}, \ \langle \ , \ \rangle$$

and almost always occur in pairs. When writing a complicated expression, you should consider whether the addition of brackets would remove any ambiguity, or simply make the expression easier to read. For example, $((6/1) - 5)/7$ is no longer ambiguous, as it can only equal $\frac{1}{7}$.