Mathematical Communication Mock Exam Answers

The following solutions are not complete, as they do not contain full working. They are intended to provide a quick indication as to whether you have solved the questions correctly. In an exam you would need to give full details.

Section A

Question 1:
A necessary condition is one that must hold for the given conclusion to occur.
A sufficient condition guarantees that the given conclusion will occur.

(a) Neither. Not necessary, e.g. \( x = 1 \) and \( y = -4 \).
Not sufficient, e.g. \( x = -1 \) and \( y = 4 \).

(b) The condition is sufficient as it rearranges to give \( y > 0 \). Not necessary, e.g. \( x = 1 \) and \( y = 0 \).

(c) Sufficient, as can divide both sides by \( x^2 \). Not necessary, e.g. \( x = 0 \) and \( y = 1 \).

Question 2:

(a) FALSE, e.g. \( A = B = C \).

(b) FALSE, e.g. \( A = \{1\}, B = \{1, 2, 3\} \) and \( C = \{3\} \).

(c) FALSE, e.g. \( A = U \) (the universe) and \( C \) is a non-empty proper subset of \( U \).

(d) TRUE, as \( B \setminus A \neq \emptyset \) is contained in \( A \cup B \) but not in \( A \cap B \).

Question 3:
The truth table is given by

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<th>( q )</th>
<th>( r )</th>
<th>( q \to r )</th>
<th>( p \lor (q \to r) )</th>
<th>( p \lor q )</th>
<th>( (p \lor q) \to r )</th>
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The columns for \( p \lor (q \to r) \) and \( (p \lor q) \to r \) are different so the propositions are inequivalent.

Question 4:

(a) The contrapositive of \( p \to q \) is \( (\neg q) \to (\neg p) \).

(b) A contradiction is a proposition which is never true, e.g. \( p \land (\neg p) \).

(c) \( p = T, q = F, r = F \), so the answers are
(i) misuse of symbols as \( s \) is not defined, (ii) misuse of symbols as brackets missing, (iii) True, (iv) False.

Question 5:

(a) This is FALSE, e.g. \( x = 1 \).

(b) This is TRUE as \( (\forall x) o(x) \) is false, and any statement \( F \to s \) is true.

(c) This is not defined as we have no predicate \( n(y) \).

(d) This should say \( d(x, y) \), in which case it is false, as for any \( x \) the value \( y = x + 2 \) does not satisfy.

Question 6:
See Coursework 4.
Section B

Question 7:
(a) See Coursework 4.
(b) Number of distinct pairs is \(\binom{n}{2}\) which is \(n(n-1)/2\). Largest possible sum is \(n^2/5 - n/2 - 2n/5 - 2 = 2n^2/5\).
(c) \(p - m = n^2/2 - n/2 - 2n/5 + 3 = n^2/10 - n/2 + 3 = n(n-5)/10 + 3 > 0\) if \(n > 5\).
(d) Suppose that every pair of numbers has a different sum. Then the number of possible sums is greater than the number of pairs, which contradicts part (c).
(e) If \(a + b = a + c\) then \(b = c\), which contradicts the two pairs being distinct.
(f) No, eg \(n = 10\) take \{1,10\} or \{2,9\} or \{3,8\} or \{4,7\}, ...
(g) No because we have no idea which numbers have been chosen.

Question 8:
(a) i. \(A \cup (B \setminus C)\).
   ii. \(((A \cup B) \cap C) \setminus (A \cap B)\).
   iii. \(A \setminus A\).
(b) (Omitted.)
(c) No, this is Russell’s paradox.
(d) See Coursework 2.

Question 9:
(a)(i) A function \(f\) is injective if \(f(x) = f(y)\) implies that \(x = y\).
(ii) A function \(f : X \rightarrow Y\) is surjective if for all \(y \in Y\) there exists \(x \in X\) with \(f(x) = y\).
(iii) A function is bijective if it is both injective and surjective.
(b) (i) \(f\) is injective, not surjective.
   (ii) \(g\) is surjective, not injective (as \(g(2) = 2 = g(4)\)).
   (iii) \(h(x)\) is odd for all \(x\) so not surjective. \(h(0) = 1 = h(-1)\) so not injective.
(c) Two sets have the same cardinality if there exists a bijection between them. A set \(X\) has cardinality \(\aleph_0\) if there exists a bijection from \(\mathbb{N}\) to \(X\).
(d) See course notes.
(e) See course notes.
(f) \((p \land \neg q) \lor (\neg p \land q)\), verify with truth table as in Q3.