Section A

Answer *all six* questions from this section. Each question carries 8 marks.

1. Give the definition of a necesssary and of a sufficient condition. For each of the following determine whether the given condition is necessary, sufficient, both, or neither. Give reasons for your answers.

   (a) Condition: $x + y > 0$ Conclusion: $x > 0$.

   (b) Condition: $x + y > x - y$ Conclusion: $y \geq 0$.

   (c) Condition: $x^2 y > x^2$ Conclusion: $y > 0$.

2. For each of the following statements determine (giving reasons) whether it is true or false. For each false statement, give an example to show that it fails.

   (a) If $A \subseteq B$ and $B \subseteq C$ then $A \subset C$.

   (b) If $A \subset B$ and $B \cap C \neq \emptyset$ then $A \cap C \neq \emptyset$.

   (c) If $A' \subset C$ then $A \subset C'$.

   (d) If $A \subset B$ then $A \cap B \neq A \cup B$.

3. By using truth tables determine whether $p \vee (q \rightarrow r)$ and $(p \vee q) \rightarrow r$ are equivalent propositions.

4. (a) Write down an expression for the contrapositive of $p \rightarrow q$.

   (b) Define what it means for a proposition to be a contradiction, and give an example involving an arbitrary proposition p.

   (c) Let p be the proposition "*2 is an even number*", q be the proposition "*Paris is not the capital of France*", and r be the proposition "*the earth is flat*". Determine whether each of the following is true, false, or misuses logical symbols.

      i. $\neg(q \vee (r \wedge s))$.

      ii. $q \vee r \rightarrow q$.

      iii. $((r \wedge p) \rightarrow q) \rightarrow p$.

      iv. $(p \rightarrow q) \wedge (r \rightarrow p)$.

5. Let $o(x)$ be the predicate "x is odd" with $x \in \mathbb{Z}$, $e(x)$ be the predicate "x is even" with $x \in \mathbb{Z}$, and $d(x, y)$ be the predicate "$x - y$ is odd" with $x, y \in \mathbb{Z}$. Determine whether each of the following is true or false, giving reasons for your answers.

   (a) $(\forall x)(e(x) \vee o(2x))$.

   (b) $((\forall x)o(x)) \rightarrow ((\exists x)e(x))$.

   (c) $(\exists x)(\exists y)(o(x) \wedge (\neg(n(y))))$.

   (d) $(\exists x)(\forall y)(q(x, y))$.

6. Prove that there are exactly 2^n subsets of each set with n elements.

Section B

Answer *two* questions from this section. Each question carries 26 marks.

7. (a) Prove for all positive integers n that

$$\sum_{j=1}^{n} j^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

[8]

Suppose we pick a set of 1000 different numbers all less than 200,000. Must this set contain 4 distinct numbers such that the sum of the first two equals the sum of the last two? This may seem unlikely, but in this question we shall prove that it is so — a special case of:

Theorem: *In any set of $n \geq 5$ distinct natural numbers all less than $n^2/5$ we can find four distinct elements such that the sum of the first two equals the sum of the last two.*

(b) Show that the number p of distinct pairs that can be chosen from our set of n objects is $p = n(n-1)/2$, and write down an expression for m, the largest possible sum of such a pair. **[3]**

(c) Show that for $n \geq 5$ we have that $p > m$. **[5]**

(d) Using the previous part, deduce that there must be two different pairs of numbers having the same sum. **[2]**

(e) It remains to show that the two pairs do not have an element in common: why can the desired two pairs not be of the form $\{a, b\}$ and $\{a, c\}$? **[3]**

(f) Is the desired set of four numbers guaranteed to be unique? **[3]**

(g) Is it possible to give a constructive proof of this result? Give reasons for your answer. **[2]**

Turn over ...

8. (a) Let $A = \{1, 2, 3, 4\}$, $B = \{2, 5, 8, 11, 14\}$ and C be the set of prime numbers. Using these three sets and the operations $\cup$, $\cap$, and $\backslash$ construct

i. $\{1, 2, 3, 4, 8, 14\}$.

ii. $\{3, 5, 11\}$.

iii. $\emptyset$. **[6]**

(b) Given X, Y and Z three sets inside some universe U, illustrate with Venn diagrams the sets

i. $(X \cap Y) \cup Z$.

ii. $(X' \backslash Y) \cap Z$. **[6]**

(c) Does the set of all sets which are not members of themselves exist? Give the popular name for this result (you do not need to give a proof). **[2]**

(d) Prove without using Venn diagrams or membership tables that

$$(X \backslash Y) \cap Z = (X \cap Z) \backslash (Y \cap Z).$$

[12]

9. (a) Define what it means for a function $f : X \to Y$ to be (i) injective; (ii) surjective; (iii) bijective. **[3]**

(b) For each of the following functions, determine whether it is injective, surjective, both, or neither. Give reasons for your answers.

i. $f : \{1, 2, 3\} \longrightarrow \{2, 4, 6, 8\}$;
$f(x) = 2x$.

ii. $g : \mathbb{N} \backslash \{1\} \longrightarrow$ {prime numbers};
$g(n) =$ smallest prime factor of n.

iii. $h : \mathbb{Z} \longrightarrow \mathbb{Z}$;
$h(x) = x^2 + x + 1$ **[10]**

(c) Define what it means for a set to have cardinality $\aleph_0$. **[2]**

(d) Write down the truth tables for $\wedge$, $\vee$ and $\rightarrow$. **[3]**

(e) State De Morgan's laws. **[4]**

(f) Find a proposition involving p and q which is true if exactly one of p and q is true, and is false otherwise. Verify your answer with a truth table. **[4]**

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