Predicate logic

So far we have looked at propositional logic. There is a more general kind of logic called *predicate logic*.

A *predicate* is either a proposition as before, or it is a statement of the form

$$p(x_1, x_2, \ldots, x_n)$$

where each x_i is a variable coming from some set D_i, such that for each choice of $x_1, \ldots, x_n$ the statement $p(x_1, \ldots, x_n)$ is a proposition. This is a little bit complicated, so we will consider some examples.

Example 3.12: (a) Let $p(x)$ be *"x is greater than 3"*, where $x \in \mathbb{Z}$.
Then $p(2)$ is false, but $p(4)$ is true.
(b) Let $q(x)$ be "$x^2 - y^2 < 0$" where $x \in \mathbb{Z}$ and $y \in \mathbb{R}$.
Then $q(1, 2)$ is "$1 - 4 < 0$" which is true, while $q(4, \sqrt{3})$ is "$16 - 3 < 0$" which is false.

Thus a predicate can be regarded as a function which for every choice of variables is either true or false. Note that predicates are only defined when the variables come from the sets given in the definition. Thus in 3.12(b) the sentence $q(\sqrt{2}, 4)$ is not defined.

We can make new predicates from old using $\vee$, $\wedge$, $\neg$, and $\rightarrow$.

Example 3.13: (a) Let $p(x)$ be "$x > 2$", where $x \in \mathbb{R}$ and $q(x)$ be "$x < 5$", where $x \in \mathbb{R}$. Then $p(x) \wedge q(x)$ is *"$x > 2$ and $x < 5$"* with $x \in \mathbb{R}$. This is true for $x = 3$ and false for $x = 1$.
(b) Let p and q be as in (a). Then $p(x) \rightarrow q(x)$ is *"if $x > 2$ then $x < 5$"* with $x \in \mathbb{R}$.
If $x = 3$ then this is true, and if $x = 6$ then this is false.

Note that when we write e.g. $p(x) \wedge q(x)$ we mean the *same* value for x in p and in q. If we write $p(x) \wedge q(y)$ then we can choose *different* values for x and y (although they can still be the same if we so choose).

Predicates are a useful way to write down certain statements, but they become much more useful in mathematics when combined with quantifiers. There are two quantifiers which we shall use.

The *universal quantifier*, written $\forall$, corresponds to the English phrase *"for all"* or *"for each"*. If $p(x)$ is a predicate with $x \in D$, then

$$(\forall x)(p(x))$$

means *"for all $x \in D$, $p(x)$ is true"*.

The *existential quantifier*, written $\exists$, corresponds to the English phrase *"there exists"* or *"for some"*. If $p(x)$ is a predicate with $x \in D$, then

$$(\exists x)(p(x))$$

means *"there exists $x \in D$ such that $p(x)$ is true"*.

Example 3.14: Let $h(x)$ mean *"x is happy"*, where x ranges over the set of mathematicians. Then

$$(\forall x)(h(x)) \text{ means } \textit{"all mathematicians are happy"},$$
$$(\exists x)(h(x)) \text{ means } \textit{"there exists a happy mathematician"}.$$

Suppose that $a(x)$ means *"x is an algebraist"*, where x runs over the set of mathematicians. Then $(\forall x)(a(x) \rightarrow h(x))$ means *"for all mathematicians x, if x is an algebraist then x is happy"*, or put more simply, *"all algebraists are happy"*.

By combining quantifiers, we can express quite complicated ideas.

Example 3.15: Let $p(x, y)$ mean *"$x - y$ is an integer"*, where x and y are real numbers. Then

1. $(\forall x)(\forall y)(p(x, y))$ means *"for all real numbers x and y, $x - y$ is an integer"* which is false (e.g take $x = 1$ and $y = 0.5$).
2. $(\exists x)(\exists y)(p(x, y))$ means *"there exist real numbers x and y such that $x - y$ is an integer"* which is true (e.g take $x = 1$ and $y = 0$).
3. $(\forall x)(\exists y)(p(x, y))$ means *"for all real numbers x there exists a real number y such that $x - y$ is an integer"* which is true (e.g take $y = x - 1$).
4. $(\exists y)(\forall x)(p(x, y))$ means *"there exists a real number y such that for all real numbers x we have that $x - y$ is an integer"* which is false (e.g. we could take $x = y - 0.5$).

Note in particular the final pair of examples — the order in which we write quantifiers is very important!

Let us see how to write down some common mathematical phrases using quantifiers. Let $p(x)$ mean *"x is an A"* and $q(x)$ mean *"x is a B"*, where in each case x runs over some set D. Then *"Every A is a B"* can be written

$$(\forall x)(p(x) \rightarrow q(x)).$$

"No A is a B" can be written

$$(\forall x)(p(x) \rightarrow (\neg q(x))).$$

"Some As are Bs" can be written

$$(\exists x)(p(x) \wedge q(x)).$$

"Some A is not a B" can be written

$$(\exists x)(p(x) \wedge (\neg q(x))).$$

We have seen how to negate the various operations of proposition logic (for "and" and "or" these identities were known as De Morgan's laws). We would also like to be able to negate a statement involving quantifiers.

Suppose that $p(x)$ is a predicate where x ranges over a set D. We have

1. $(\forall x)(p(x))$ meaning *"all x have property p"*.
2. $(\exists x)(\neg p(x))$ meaning *"some x does not have property p"*.
3. $(\exists x)(p(x))$ meaning *"some x has property p"*.
4. $(\forall x)(\neg p(x))$ meaning *"no x has property p"*.

Now (2) is the denial of (1), and so

$$\neg((\forall x)(p(x))) \text{ is the same as } (\exists x)(\neg p(x)).$$

Also (4) is the denial of (3), and so

$$\neg((\exists x)(p(x))) \text{ is the same as } (\forall x)(\neg p(x)).$$

Example 3.16: (a) If $p(x)$ means *"x is positive"* where $x \in \mathbb{Z}$, then $\neg((\forall x)(p(x)))$ means *"not all integers are positive"*, which is the same as saying *"there exists an integer which is not positive"*, i.e. $(\exists x)(\neg p(x))$.

(b) Similarly, if $q(x)$ means *"x is even"* where x runs over the set of prime numbers then $\neg((\exists x)(p(x)))$ means *"there does not exist an even prime number"*, which is the same as saying *"all prime numbers are odd"*, i.e. $(\forall x)(\neg p(x))$.