

Section A

Answer *all six* questions from this section. Each question carries 8 marks.

1. Give the definition of a necessary and of a sufficient condition. For each of the following determine whether the given condition is necessary, sufficient, both, or neither. Give reasons for your answers.
 - (a) Condition: $x < y$ Conclusion: $x^2 + y^2 - 2xy > 0$.
 - (b) Condition: $x^2y > 0$ Conclusion: $xy > 0$.
 - (c) Condition: $x^2y^2 > 0$ Conclusion: $xy > 0$.
2. For each of the following statements determine (giving reasons) whether it is true or false. For each false statement, give an example to show that it fails.
 - (a) If $A \subset B$ and $B \subseteq C$ then $A \subseteq C$.
 - (b) If $A \subset C$ and $B \subseteq C$ then $A \cap B \neq \emptyset$.
 - (c) If $A \subseteq B$ and $A' \cap B \neq \emptyset$ then $A \subset B$.
 - (d) If $A \neq B$ then $(A \setminus B) \cup B \neq B$.
3. By using truth tables determine whether $p \wedge (q \vee r)$ and $(p \wedge q) \vee r$ are equivalent propositions.
4.
 - (a) Write down an expression for the contrapositive of $p \rightarrow q$.
 - (b) Define what it means for a proposition to be a tautology, and give an example involving an arbitrary proposition p .
 - (c) Let p be the proposition “*2 is an odd number*”, q be the proposition “*Paris is the capital of France*”, and r be the proposition “*the earth is flat*”. Determine whether each of the following is true, false, or misuses logical symbols.
 - i. $\neg(q \rightarrow r)$.
 - ii. $((q \vee p) \rightarrow r) \rightarrow r$.
 - iii. $\neg q \vee (r \rightarrow p)$.
 - iv. $(p \rightarrow p) \wedge (r \rightarrow q)$.

Turn over ...

5. Let $p(x)$ be the predicate “ $x > 0$ ” with $x \in \mathbb{Z}$, $n(x)$ be the predicate “ $x < 0$ ” with $x \in \mathbb{Z}$, and $q(x, y)$ be the predicate “ $x > y$ ” with $x, y \in \mathbb{Z}$. Determine whether each of the following is true or false, giving reasons for your answers.

(a) $(\forall x)(p(x) \vee n(x))$.

(b) $((\forall x)p(x)) \rightarrow ((\exists x)n(x))$.

(c) $(\exists x)(\exists y)(p(x) \wedge n(y))$.

(d) $(\exists x)(\forall y)(q(x, y))$.

6. Prove that the square of every odd number is of the form $8a + 1$ for some $a \in \mathbb{N}$.

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Section B

Answer *two* questions from this section. Each question carries 26 marks.

7. (a) Prove for all positive integers n that 8 divides $7^{2n-1} + 9^n$. [8]

In the remainder of this question we will investigate some of the properties of rational and irrational numbers. Recall that a real number x is *rational* if it can be written in the form $x = a/b$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$, and is *irrational* otherwise.

- (b) Prove that if x is irrational then \sqrt{x} is irrational by considering the contrapositive. [4]

- (c) Is

$$\frac{\sqrt{2}}{1 + \sqrt{2}}$$

rational or irrational? Give reasons for your answer (you may assume that $\sqrt{2}$ is irrational). [5]

- (d) Which two of the following are true? Identify which can be proved directly, and which can be proved by contradiction. Give proofs or counterexamples in each case.
- If x and y are rational then $x + y$ is rational.
 - If x is rational and y is irrational then $x + y$ is irrational.
 - If x and y are irrational then $x + y$ is irrational.
- (Hint: you may want to use your answer to part (ii).) [9]

Turn over ...

