Mathematical Communication Answers, Summer 2007

Section A

Question 1:

A necessary condition is one that must hold for the given conclusion to occur.
A sufficient condition guarantees that the given conclusion will occur. **[2]**

(a) The condition is sufficient as if $x < y$ then $(x - y)^2 > 0$.
The condition is not necessary, e.g. $x = 3$ and $y = 2$. **[2]**

(b) The condition is not sufficient, e.g. $x = -1$ and $y = 1$.
The condition is not necessary, e.g. $x = -1$ and $y = -1$. **[2]**

(c) The condition is necessary as $x^2y^2 \not> 0$ implies that $xy = 0$.
The condition is not sufficient, e.g. $x = 1$ and $y = -1$. **[2]**

Question 2:

(a) TRUE. If $x \in A$ then $x \in B$, and if $x \in B$ then $x \in C$.
Therefore if $x \in A$ then $x \in C$, so $A \subseteq C$. **[2]**

(b) FALSE. E.g. take $A = \{1\}$, $B = \{2\}$ and $C = \{1, 2\}$. **[2]**

(c) TRUE. If $x \in A$ then $x \in B$. Also, there exists $x \notin A$ with $x \in B$.
Therefore $A \subset B$. **[2]**

(d) FALSE. E.g. take $A = \{1\}$ and $B = \{1, 2\}$. **[2]**

Question 3:

The truth table is given by

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$(p \wedge q) \vee r$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	F	T
T	F	F	F	F	F	F
F	T	T	T	F	F	T
F	T	F	T	F	F	F
F	F	T	T	F	F	T
F	F	F	F	F	F	F

[6]

The columns for $p \wedge (q \vee r)$ and $(p \wedge q) \vee r)$ differ in rows 5 and 7, so the propositions are inequivalent. **[2]**

Question 4:

(a) The contrapositive of $p \to q$ is $(\neg q) \to (\neg p)$. **[2]**

(b) A tautology is a proposition which is always true, e.g. $p \vee (\neg p)$. **[2]**

(c) $p = F$, $q = T$, $r = F$, so the answers are
(i) T, (ii) T, (iii) Misuse of symbols, (iv) T. **[4]**

Question 5:

(a) This is FALSE as $x = 0$ satisfies neither $x > 0$ nor $x < 0$. **[2]**

(b) This is TRUE as $(\forall x)p(x)$ is false, and any statement $F \to s$ is true. **[2]**

(c) This is TRUE, e.g. with $x = 1$ and $y = -1$. **[2]**

(d) This is FALSE, as for any x the value $y = x + 1$ does not satisfy $x > y$. **[2]**

Question 6:

Any odd number n is of the form $n = 2l + 1$ for some $l \in \mathbb{Z}$. Therefore

$$n^2 = (2l + 1)^2 = 4l^2 + 4l + 1 = 4(l^2 + l) + 1.$$

Thus it is enough to show that $l^2 + l$ is even. **[3]**
If l is even then $l = 2m$ for some $m \in \mathbb{Z}$, so

$$l^2 + l = 4m^2 + 2m = 2(2m^2 + m)$$

is divisible by 2. If l is odd then $l = 2m + 1$ for some $m \in \mathbb{Z}$, so

$$l^2 + l = 4m^2 + 4m + 1 + 2m + 1 = 2(2m^2 + 3m + 1)$$

is also divisible by 2. Thus $l^2 + l$ is always even, and so n^2 is of the form $8a + 1$ for some $a \in \mathbb{Z}$. But $n^2 \geq 1$, so $a \in \mathbb{N}$. **[5]**

Section B

Question 7:

(a) We will proceed by induction on n. Let $P(n)$ be the proposition "8 divides $7^{2n-1} + 9^n$".
The statement $P(1)$ is "8 divides $7+9$" which is true. So it remains to show that if $P(k)$ is true then this implies that $P(k+1)$ is true, for all $k \in \mathbb{N}$. **[3]**

Consider $P(k+1)$. We have that

$$7^{2(k+1)-1} + 9^{k+1} = 49 \times 7^{2k-1} + 9 \times 9^k = 9(7^{2k-1} + 9^k) + 40 \times 7^{2k-1}.$$

By the inductive hypothesis the bracketted term is divisible by 8. Clearly the remaining term is divisible by 8 (as it is divisible by 40). Therefore the whole expression is divisible by 8, and so $P(k)$ true implies that $P(k+1)$ is true. The result now follows by induction. **[5]**

(b) The contrapositive says that if $\sqrt{x}$ is rational then x is rational. This is equivalent to the original statement, so it is enough to prove that this is true. To prove this, suppose that $\sqrt{x}$ is rational. Then

$$\sqrt{x} = \frac{a}{b}$$

for some $a, b \in \mathbb{N}$ with $b \neq 0$. Therefore

$$x = \frac{a^2}{b^2}$$

which is clearly rational, as a^2 and b^2 are integers with $b^2 \neq 0$. **[4]**

(c) Suppose that the given fraction is rational, i.e. that

$$\frac{\sqrt{2}}{1+\sqrt{2}} = \frac{a}{b}$$

with $a, b \in \mathbb{Z}$ and $b \neq 0$. Then

$$b\sqrt{2} = a + a\sqrt{2}$$

and so

$$\sqrt{2} = \frac{a}{b-a}$$

is rational. But this contradicts the fact that $\sqrt{2}$ is irrational, and so the given fraction is irrational. **[5]**

(d) (i) (Direct.) Let $x = \frac{a}{b}$ and $y = \frac{c}{d}$ with $a, b, c, d \in \mathbb{Z}$ Then

$$x + y = \frac{ad + bc}{cd}$$

is clearly rational. **[3]**

(ii) (By contradiction.) Suppose that $x = \frac{a}{b}$ and $x + y = \frac{c}{d}$ are rational with y irrational. Then

$$y = (x+y) - x = \frac{c}{d} - \frac{a}{b} = \frac{bc - ad}{bd}$$

is rational, which is impossible. Therefore $x + y$ is irrational. **[3]**

(iii) (False.) By part (ii) $a + b$ is irrational if a is rational and b irrational. If $x = -b$ and $y = a + b$, both irrational, then $x + y = a$ is rational. **[3]**

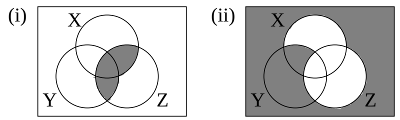
Question 8:

(a) (i) $\{3, 5\} = A \cap B$.

(ii) $\{3, 4, 5\} = (B \cup C) \cap A$.

(iii) $\emptyset = A \backslash A$ (for example). **[6]**

(b)



[6]

(c) The subsets are

$$\emptyset, \{1\}, \{2\}, \{\{3,4\}\}, \{1,2\}, \{1,\{3,4\}\}, \{2,\{3,4\}\}, \{1,2,\{3,4\}\}.$$

[3]

(d) Let $E = (X \cup Y) \cap Z$ and $F = (X \cap Z) \cup (Y \cap Z)$. To show that $E = F$ it is enough to show that $E \subseteq F$ and $F \subseteq E$. **[2]**

First suppose that $x \in E$. Then $x \in X \cup Y$ and $x \in Z$. But this means that either $x \in X$ and $x \in Z$ or $x \in Y$ and $x \in Z$. Therefore $x \in (X \cap Z)$ or $x \in (Y \cap Z)$, i.e. $x \in F$. We have shown that if $x \in E$ then $x \in F$, and hence that $E \subseteq F$. **[4]**

Next suppose that $x \in F$. Then $x \in X \cap Z$ or $x \in Y \cap Z$. But this means that either $x \in X$ and $x \in Z$ or $x \in Y$ and $x \in Z$. In either case we have that $x \in Z$, and hence we have that $x \in Z$ and either $x \in X$ or $x \in Y$. But this is the same as saying that $x \in E$. We have shown that if $x \in F$ then $x \in E$, and hence that $F \subseteq E$. As $E \subseteq F$ and $F \subseteq E$ we have that $E = F$ as required. **[5]**

Question 9:

(a)(i) A function f is injective if $f(x) = f(y)$ implies that $x = y$.

(ii) A function $f : X \longrightarrow Y$ is surjective if for all $y \in Y$ there exists $x \in X$ with $f(x) = y$.

(iii) A function is bijective if it is both injective and surjective. **[3]**

(b) (i) f is injective, as $f(x) = f(y)$ implies that $7 - x = 7 - y$, and hence that $x = y$. But $f(x) \leq 7$ for all $x \in \mathbb{N}$, and hence f is not surjective. **[3]**

(ii) g is not injective as $g(0) = 0 = g(-1)$. Also, g is not surjective as g only produces even answers. **[3]**

(iii) h is injective as $h(\frac{a}{b}) = h(\frac{c}{d})$ implies that

$$\frac{1}{\frac{a}{b}+1} = \frac{1}{\frac{c}{d}+1}$$

and hence

$$\frac{a}{b} + 1 = \frac{c}{d} + 1$$

which gives that $\frac{a}{b} = \frac{c}{d}$. Also h is surjective, as if $y \in \mathbb{Q}$ with $y \neq 0$ then

$$\frac{1}{x+1} = y$$

has solution $x = \frac{1}{y} - 1 \neq -1$. Therefore h is bijective. **[4]**

(c) Two sets have the same cardinality if there exists a bijection between them. A set X has cardinality $\aleph_0$ if there exists a bijection from $\mathbb{N}$ to X. **[2]**

(d) That the set of real numbers [or just the set of real numbers strictly between 0 and 1] does not have cardinality $\aleph_0$. **[2]**

(e) Consider the function $f : \mathbb{N} \longrightarrow \{x^3 : x \in \mathbb{Z}\}$ given by

$$f(x) = \begin{cases} n^3 & \text{if } x \text{ is even, with } x = 2n \\ -n^3 & \text{if } x \text{ is odd, with } x = 2n+1 \end{cases}$$

[3]

This is clearly injective and surjective, and hence gives a bijection from $\mathbb{N}$ to $\{x^3 : x \in \mathbb{Z}\}$. Therefore $\{x^3 : x \in \mathbb{Z}\}$ has cardinality $\aleph_0$. **[3]**

(f) No, the cardinality of a finite set is determined by the number of elements in it. **[3]**